Design of Cognitive Radio Systems Under Temperature-Interference Constraints: A Variational Inequality Approach

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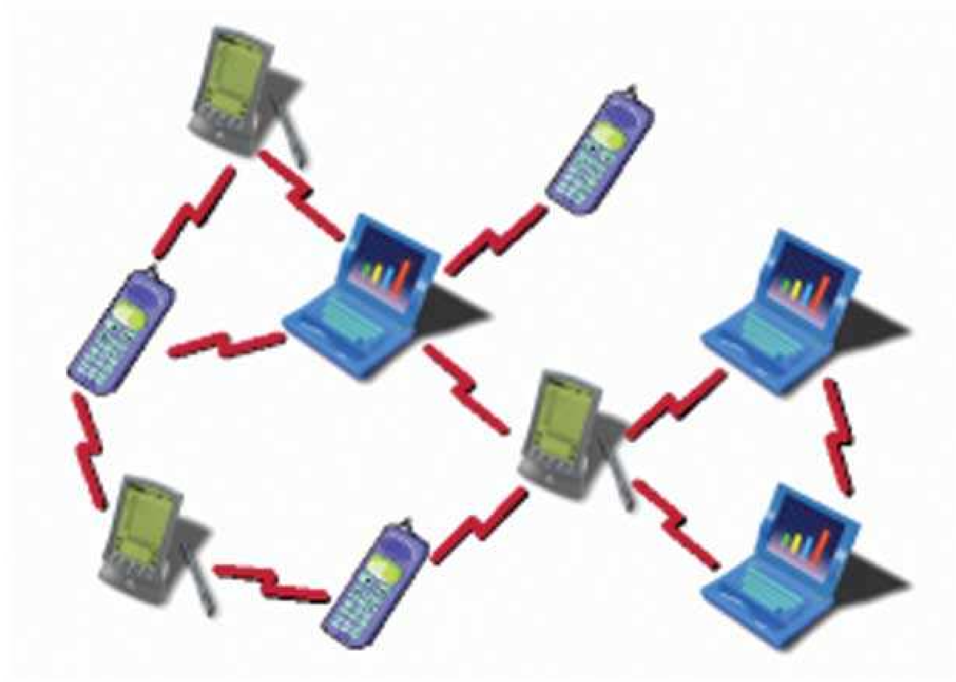
Outline of the Talk

- Games in Ad-Hoc Networks
  - the waterfilling operator and the iterative waterfilling algorithm
- Games in Cognitive Radio Systems
  - Individual temperature-interference constraints
  - Global temperature-interference constraints
- Variational Inequality Theory
- Numerical results
- Summary

Games in Ad-Hoc Networks

Competitive Ad-Hoc Networks

- Consider a decentralized and competitive network of users fighting for the resources (i.e., spectrum):



- Game theory is a very adequate mathematical framework to analyze such systems and design effective strategies.

Signal Model for Ad-Hoc Networks

- Signal received by link q, for carriers $k = 1, \ldots, N$:

$$y_{qq}(k) = H_{qq}(k)\, x_q(k) + \sum_{r \neq q} H_{qr}(k)\, x_r(k) + w_q(k).$$

- The optimization variables correspond to the power allocation over the carriers: $\mathbf{p}_q = \{p_q(k)\}_{k=1}^{N}$.

- There is a power budget for each user:

$$\sum_{k=1}^{N} p_q(k) \leq P_q.$$

- The payoff for user q is the transmission rate:

$$R_q\left(\mathbf{p}_q, \mathbf{p}_{-q}\right) = \sum_{k=1}^{N} \log\left(1 + \mathsf{sinr}_q(k)\right)$$

where

$$\mathsf{sinr}_q(k) = \frac{\left|H_{qq}(k)\right|^2 p_q(k)}{1 + \sum_{r \neq q} \left|H_{rq}(k)\right|^2 p_r(k)}.$$

- The feasible set for the variables is

$$\mathcal{P}_q = \left\{ \mathbf{p}_q \in \mathbb{R}_+^N : \sum_{k=1}^{N} p_q(k) = P_q \right\}.$$

Game Formulation for Ad-Hoc Networks

- Each of the Q users selfishly maximizes its own rate subject to the constraints:

$$\begin{array}{ll} \underset{\mathbf{p}_q}{\text{maximize}} & \sum_{k=1}^{N} \log\left(1+\mathsf{sinr}_q(k)\right) \\ \text{subject to} & \mathbf{p}_q \in \mathcal{P}_q \end{array} \qquad q = 1, \ldots, Q.$$

- The best response for each user has a nice and simple closed-form expression:

$$\mathbf{p}_q^{\star} = \mathsf{wf}_q\left(\mathbf{p}_{-q}\right)$$

where $\mathsf{wf}_q\left(\mathbf{p}_{-q}\right) \triangleq \left(\mu_q - \mathbf{interf}_q\right)^{+}$ is the waterfilling operator.

- Different researchers have actively worked on this problem since 2001 [Yu-Ginis-Cioffi'01].

- The Nash Equilibrium (NE) is a simultaneous waterfilling for all users:

$$\mathbf{p}_q^\star = \mathsf{wf}_q\left(\mathbf{p}_{-q}^\star\right) \qquad q = 1, \ldots, Q.$$

- Convex game $\Rightarrow$ the existence of a NE follows readily.

- We can devise different iterative algorithms based on the waterfilling best response.

- New interpretation of waterfilling as a projection [ScuPalBar'06]: $\mathbf{p_q} = \left[-\mathbf{interf}_q\right]_{\mathcal{P}_q}$.

- With this new interpretation, it is simple to show uniqueness of NE and prove convergence of iterative waterfilling algorithms (IWFA).

Asynchronous IWFA

- Users update the power allocation in a totally asynchronous way based on $\mathrm{wf}_q(\cdot)$:
  - users may update at arbitrary and different times and more or less frequently than others
  - users may use an outdated measure of interference

- **Theorem [ScuPalBar'06]:** The asynchronous IWFA converges to the unique NE if $\rho(\mathbf{H}^{\max}) < 1$, where

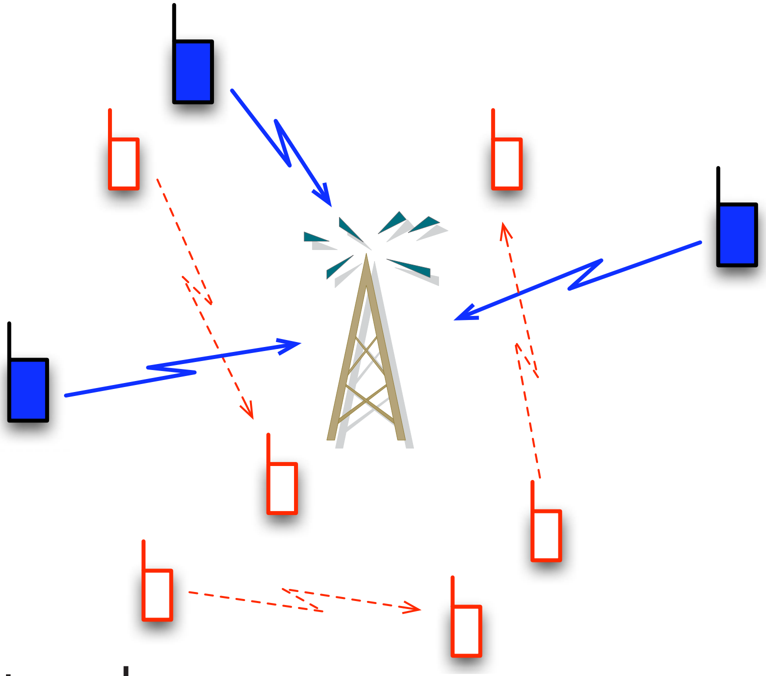
$$[\mathbf{H}^{\max}]_{qr} \triangleq \begin{cases} \max\limits_{k \in \mathcal{D}_r \cap \mathcal{D}_q} \left\{ \dfrac{|H_{rq}(k)|^2}{|H_{qq}(k)|^2} \right\} & \text{if } q \neq r \\ 0 & \text{otherwise.} \end{cases}$$

and $\mathcal{D}_q = \{1, \ldots, N\} - \{\text{bad subcarriers}\}$.

Games in Cognitive Radio (CR) Systems

CR Systems

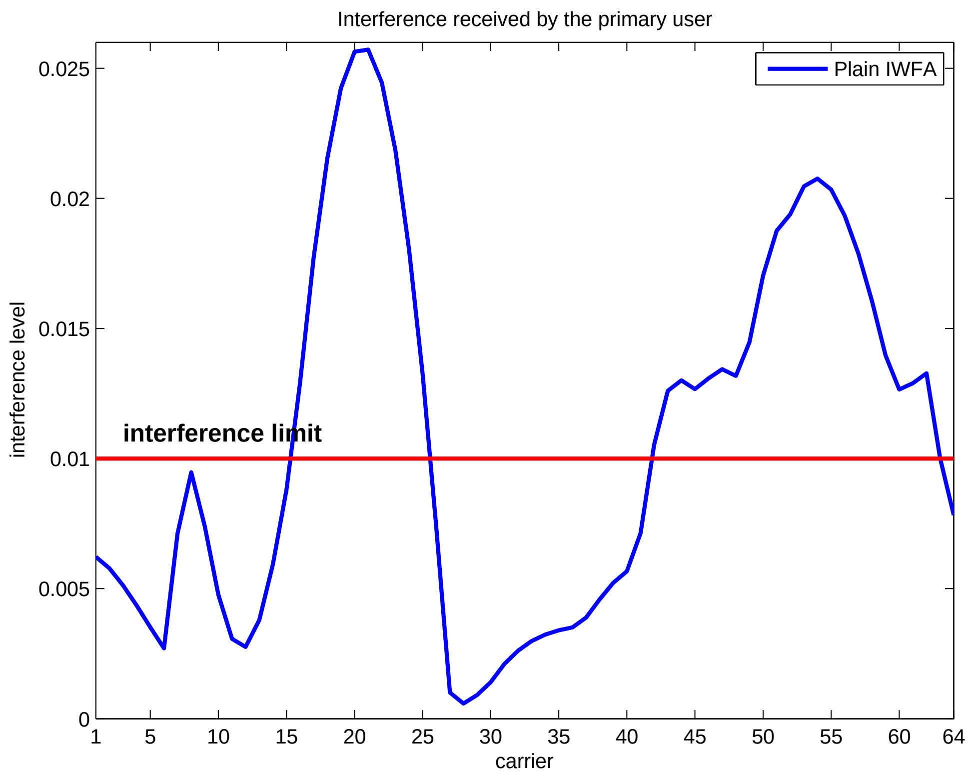
- Consider now an established network of primary users on top of which some secondary users play the previous game.



- Hierarchical CR networks
  - PU=Primary users (legacy spectrum holders)
  - SU=Secondary users (unlicensed users)

- Opportunistic communications: SUs can use the spectrum subject to not inducing too much interference on the PUs.

- The previous iterative waterfilling algorihm does not work because it violates the interference constraint:



Signal Model for CR Systems

- Same signal model as for ad-hoc networks with the additional temperature-interference constraints:

$$\left|G_{qp}(k)\right|^2 p_q(k) \leq \alpha_p$$

  where $\left|G_{qp}(k)\right|^2$ is the cross-channel gain between the qth secondary and the pth primary user and α_p is the maximum level of interference tolerable by the primary user from each secondary user.

- Equivalently, we can write these constraints as

$$p_q(k) \leq p_q^{\max}(k) \quad \forall k, q.$$

Game Formulation for Ad-Hoc Networks

- Each of the Q users selfishly maximizes its own rate subject to the constraints:

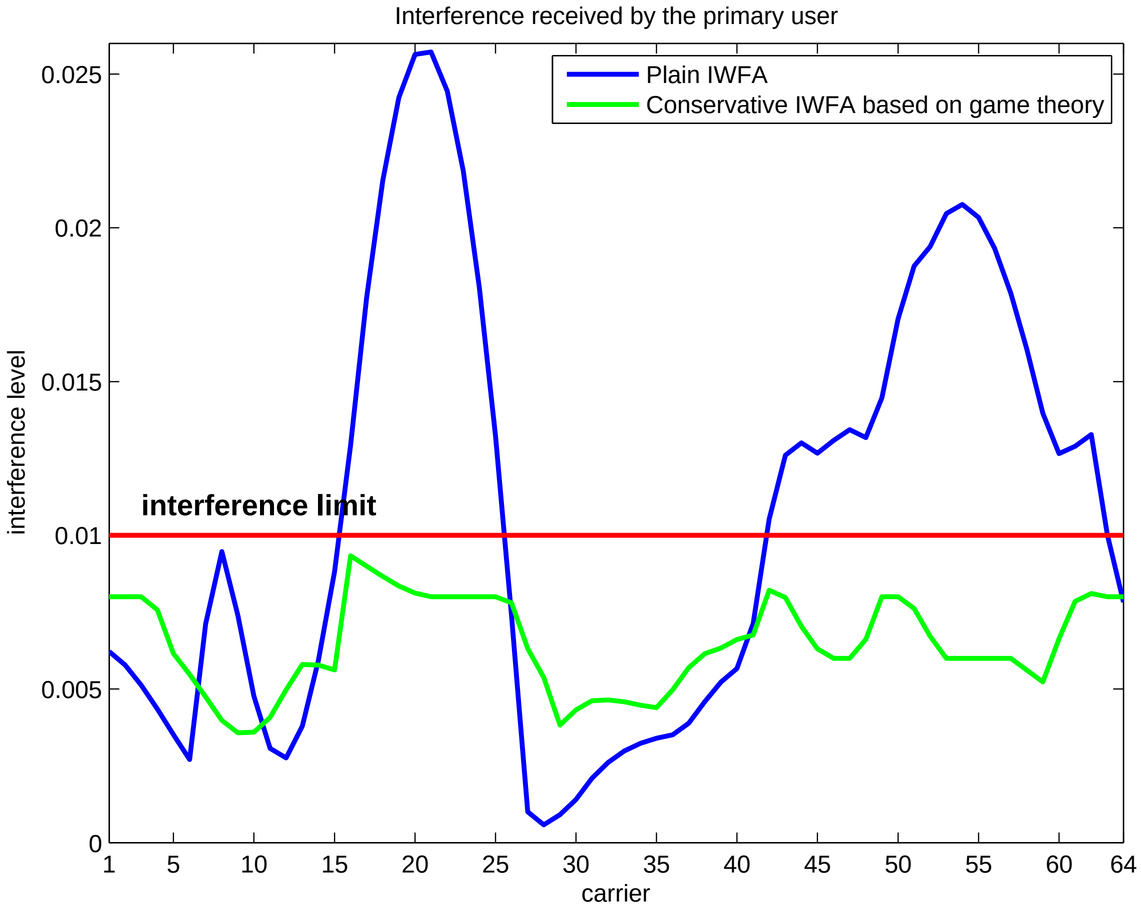
$$\begin{array}{lll} \underset{\mathbf{p}_q}{\text{maximize}} & \sum_{k=1}^{N} \log\left(1+\mathsf{sinr}_q(k)\right) & \\ \text{subject to} & \sum_{k=1}^{N} p_q(k) \le P_q & \forall q = 1,\ldots,Q. \\ & 0 \le p_q(k) \le p_q^{\max}(k), \quad \forall k & \end{array}$$

- The best response in this case also has a nice and simple closed-form expression based on a modified waterfilling with clipping from above:

$$\widetilde{\mathsf{wf}}_q\left(\mathbf{p}_{-q}\right) \triangleq \left[\mu_q - \mathbf{interf}_q\right]_{\mathbf{0}}^{\mathbf{p}^{\max}}.$$

- The analysis of this game and the algorithms are similar to the previous one.

- However, this method may be too conservative as the level of interference from each secondary user is limited individually in a conservative way:



Revised Signal Model for CR Systems

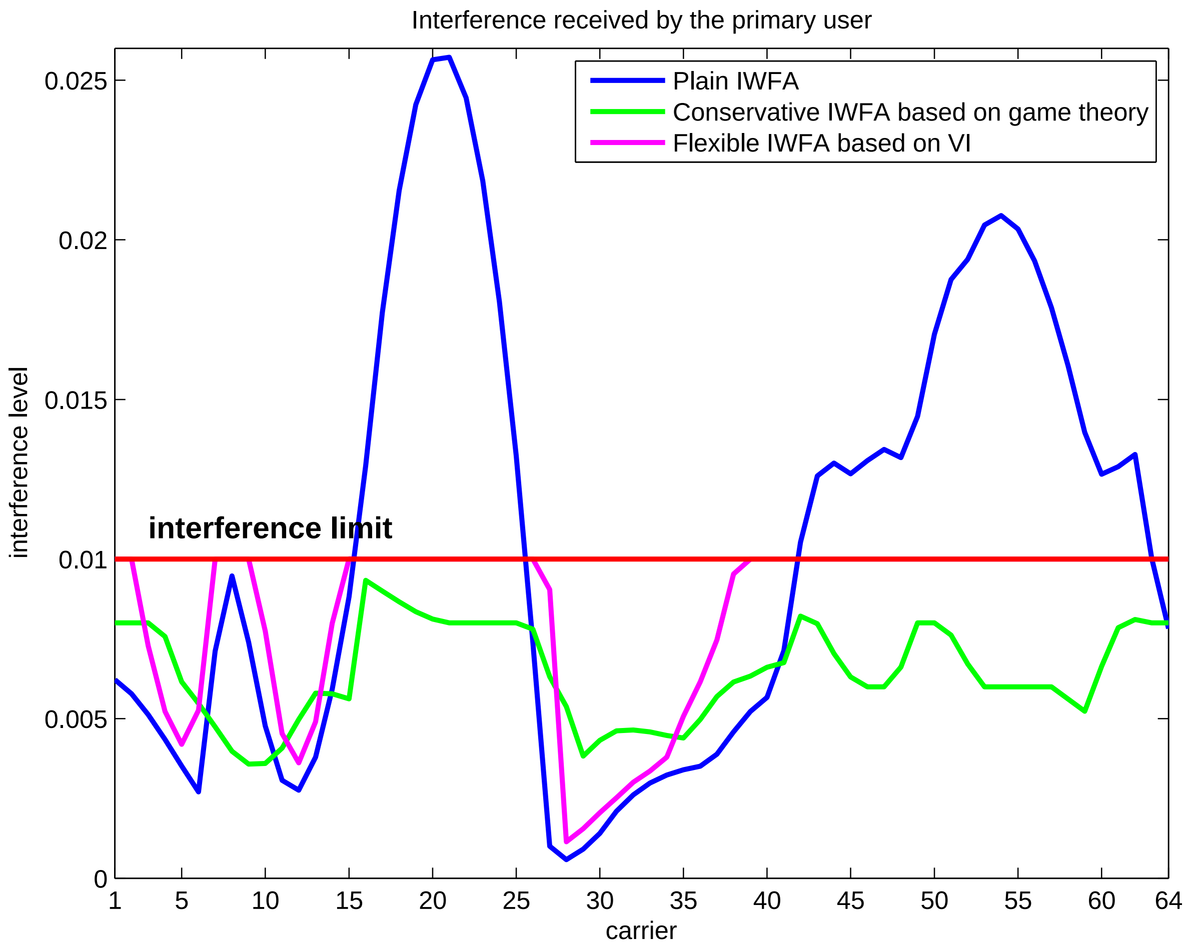
- The really important quantity is not the individual interference generated by each secondary user but the aggregate interference generated by all of them.

- We can then limit the aggregate interference instead:

$$\sum_{q=1}^{Q} |G_{qp}(k)|^2 \, p_q(k) \leq \alpha_p .$$

- This will achieve our goal without being conservative or accidentally violating the limit.

- Indeed, this new reformulation achieves our goal in a remarkable way:



- The price to pay for including the coupling constraints is twofold:
  - on a mathematical level, it complicates the analysis of the game and its design
  - on the practical side, this new game must include some mechanism to calculate the aggregate interference.

- For the practical aspect, one approach is to allow the primary users to estimate the aggregate interference and then broadcast a minimum of signaling. But this only makes sense if the signaling is really minimum and scales nicely.

- For the mathematical analysis and design, we need more advance tools: variational inequality theory.

Variational Inequality (VI) Theory

The Variational Inequality Problem

- Given a closed and convex set $\mathcal{K} \subseteq \mathbb{R}^n$ and a mapping $\mathbf{F} : \mathcal{K} \to \mathbb{R}^n$, the VI problem VI$(\mathcal{K}, \mathbf{F})$ is to find a vector $\mathbf{x}^\star \in \mathcal{K}$ such that

$$(\mathbf{y} - \mathbf{x}^\star)^T \mathbf{F}(\mathbf{x}^\star) \geq 0 \qquad \forall \mathbf{y} \in \mathcal{K}.$$

- The VI$(\mathcal{K}, \mathbf{F})$ reduces to the minimum principle (convex optimization problems) if $\mathbf{F} = \nabla f$

$$\text{VI}(\mathcal{K}, \nabla f) \quad \Longleftrightarrow \quad \min_{\mathbf{x} \in \mathcal{K}} f(\mathbf{x})$$

- The VI$(\mathcal{K}, \mathbf{F})$ encompasses a wider range of problems than classical optimization whenever $\mathbf{F} \neq \nabla f$ ($\Leftrightarrow$ $\mathbf{F}$ has not a symmetric Jacobian).

- Other classical problems that fall in the VI framework:
  - *System of equations*: $\mathcal{K} = \mathbb{R}^n \Rightarrow \mathsf{VI}(\mathcal{K}, \mathbf{F}) \Leftrightarrow \mathbf{F}(\mathbf{x}) = \mathbf{0}$
  - *Nonlinear complementarity problem (NCP):* $\mathcal{K} = \mathbb{R}^n_+ \Rightarrow \mathrm{VI}(\mathcal{K}, \mathbf{F}) \Leftrightarrow \mathrm{NCP}(\mathbf{F})$

$$\mathsf{NCP}(\mathbf{F}): \qquad \mathbf{0} \le \mathbf{x}^\star \perp \mathbf{F}(\mathbf{x}^\star) \ge \mathbf{0}$$

    (The NCP is a unifying mathematical framework for linear programming, quadratic programming, and bi-matrix games.)
  - *Fixed-point problems*: $\mathbf{F} = \mathbf{x} - \mathbf{G}(x) \Rightarrow \mathsf{VI}(\mathcal{K}, \mathbf{F}) \Leftrightarrow \mathcal{K} \ni \mathbf{x} = \mathbf{G}(\mathbf{x})$
  - *Nash equilibrium problems* (NEP): $\mathcal{K} = \prod_q \mathcal{K}_q$ and $\mathbf{F}(\mathbf{x}) = (\mathbf{F}_q(\mathbf{x}))_{q=1}^Q$ with $\mathbf{F}_q = \nabla_{\mathbf{x}_q} f_q(\mathbf{x}_q, \mathbf{x}_{-q}) \Rightarrow$

$$\mathsf{VI}(\mathcal{K}, \mathbf{F}) \quad \Longleftrightarrow \quad \min_{\mathbf{x}_q \in \mathcal{K}_q} f_q(\mathbf{x}_q, \mathbf{x}_{-q}), \qquad \forall q = 1, \ldots, Q.$$

Formulation of the Game with Coupling Constraint

- Recall the game formulation with the coupling constraint:

$$
\mathcal{G}: \quad \begin{array}{ll} \underset{\mathbf{p}_q \geq \mathbf{0}}{\text{maximize}} & \sum_{k=1}^{N} \log\left(1 + \mathsf{sinr}_q(k)\right) \\ \text{subject to} & \sum_{k=1}^{N} p_q(k) \leq P_q \end{array} \qquad \forall q = 1, \ldots, Q
$$

$$
\sum_{q=1}^{Q} \left|G_{qp}(k)\right|^2 p_q(k) \leq \alpha_p \qquad \forall p = 1, \ldots, P.
$$

- This is a GNEP with a common constraint.

- It can be "rewritten" as a VI problem (caveat: only the variational solutions with common multipliers are considered).

Formulation of the Game with Coupling Constraint via Pricing

- To deal with coupling interference constraints while keeping the optimization as decentralized as possible we introduce pricing:

$$\mathcal{G}_{\boldsymbol{\lambda}}: \quad \begin{array}{ll} \underset{\mathbf{p}_q \geq \mathbf{0}}{\text{maximize}} & r_q(\mathbf{p}_q, \mathbf{p}_{-q}) - \sum_{p=1}^{P} \sum_{k=1}^{N} \lambda_{p,k} \left|G_{qp}(k)\right|^2 p_q(k) \\ \text{subject to} & \sum_{k=1}^{N} p_q(k) \leq P_q \end{array} \qquad \forall q$$

where the prices $\lambda_{p,k} \geq 0$ are chosen such that

$$\text{(CC)}: \qquad 0 \leq \lambda_{p,k} \quad \perp \quad \alpha_{p,k} - \sum_{q=1}^{Q} \left|G_{qp}(k)\right|^2 p_q(k) \geq 0 \qquad \forall p, \quad \forall k$$

- We will now rewrite the game $\mathcal{G} \triangleq \mathcal{G}_{\boldsymbol{\lambda}} \cup \text{(CC)}$ as a VI problem.

- **Theorem 1 (Game as a VI):** *The game* $\mathcal{G}$ *is equivalent to the* VI$(\mathcal{K}, \mathbf{F})$ *where*

$$\mathcal{K} \triangleq \left\{ \mathbf{p} \in \mathbb{R}_+^{NQ} \,:\, \sum_{k=1}^{N} p_q(k) \leq P_q \text{ and } \sum_{q=1}^{Q} \left|G_{qp}(k)\right|^2 p_q(k) \leq \alpha_{p,k},\ \forall\, q,\, p\,,\, k \right\}$$

and

$$\mathbf{F}(\mathbf{p}) \triangleq \begin{pmatrix} -\nabla_{\mathbf{p}_1} r_1(\mathbf{p}) \\ \vdots \\ -\nabla r_{\mathbf{p}_Q}(\mathbf{p}) \end{pmatrix}.$$

- The equivalence is in *the following sense:*
  - *If* $\mathbf{p}^{\mathrm{VI}}$ *is a solution of the* VI, *then there exist multipliers* $\boldsymbol{\lambda}^{\mathrm{VI}}$ *such that* $(\mathbf{p}^{\mathrm{VI}}, \boldsymbol{\lambda}^{\mathrm{VI}})$ *is an equilibrium pair of* $\mathcal{G}$
  - *Conversely, if* $(\mathbf{p}^{\mathrm{NE}}, \boldsymbol{\lambda}^{\mathrm{NE}})$ *is an equilibrium pair of* $\mathcal{G}$, *then* $\mathbf{p}^{\mathrm{NE}}$ *is a solution of the* VI, *and* $\boldsymbol{\lambda}^{\mathrm{NE}}$ *are multipliers of the* VI.

Results from the VI Framework: Solution Analysis

- Using the VI framework we can study existence/uniqueness of the solution and devising distributed algorithms.

- Let's introduce now the interference violation function $\boldsymbol{\Phi}(\boldsymbol{\lambda}) : \mathbb{R}_+^{PN} \ni \boldsymbol{\lambda} \mapsto \mathbb{R}^{PN}$

$$\boldsymbol{\Phi} : \boldsymbol{\lambda} \mapsto \left(\left(\alpha_{p,k} - \sum_{q=1}^{Q} \left| G_{qp}(k) \right|^2 p_q^\star \left(k; \boldsymbol{\lambda} \right) \right)_{k=1}^{N} \right)_{p=1}^{P} \qquad (1)$$

where $\mathbf{p}^\star(\boldsymbol{\lambda}) = \left((p_q^\star(k; \boldsymbol{\lambda}))_{k=1}^{N} \right)_{q=1}^{Q}$ is a Nash equilibrium of $\mathcal{G}_{\boldsymbol{\lambda}}$ for a *given* $\boldsymbol{\lambda} = \left((\lambda_{p,k})_{k=1}^{N} \right)_{p=1}^{P}$.

- **Theorem 2 (Existence and Uniqueness of the NE of $\mathcal{G}$):**
  - *The* VI$(\mathcal{K}, \mathbf{F})$ *always admits a solution* $\mathbf{p}^{\mathrm{VI}}$ (the NE of $\mathcal{G}$)
  - *If* $\boldsymbol{\Upsilon} \succ \mathbf{0}$ *then*
    * $\mathbf{p}^{\mathrm{VI}}$ *is unique and* $\mathcal{G}$ *is equivalent to the NCP in the price* $\boldsymbol{\lambda}$

$$\text{NCP}(\boldsymbol{\Phi}): \qquad 0 \leq \boldsymbol{\lambda} \perp \boldsymbol{\Phi}(\boldsymbol{\lambda}) \geq 0$$

    * the game $\mathcal{G}$ has a unique least-norm price tuple $\boldsymbol{\lambda}^{\mathrm{NE,lm}}$.

- The uniqueness is only in the powers but not in the prices. However, under $\boldsymbol{\Upsilon} \succ \mathbf{0}$, *all* the optimal prices (=solutions to NCP) yield the same *unique* optimal powers $\mathbf{p}^{\mathrm{VI}}$.

- The NCP reformulation is instrumental to devise distributed algorithms.

Distributed Algorithms based on VI

Algorithm 1: Projection algorithm with constant step-size

`(S.0)` : Choose any $\boldsymbol{\lambda}^{(0)} \geq \mathbf{0}$, and the step size $\tau > 0$, and set $n = 0$

`(S.1)` : If $\boldsymbol{\lambda}^{(n)}$ satisfies a suitable termination criterion: `STOP`

`(S.2)` : Given $\boldsymbol{\lambda}^{(n)}$, compute $\mathbf{p}^{\star}(\boldsymbol{\lambda}^{(n)})$ as the NE solution of the NEP $\mathcal{G}_{\boldsymbol{\lambda}}$ with *fixed* prices $\boldsymbol{\lambda} = \boldsymbol{\lambda}^{(n)}$

`(S.3)` : Update the price vectors:

$$\boldsymbol{\lambda}^{(n+1)} = \left[\boldsymbol{\lambda}^{(n)} - \tau\,\boldsymbol{\Phi}\left(\boldsymbol{\lambda}^{(n)}\right)\right]^{+} \tag{2}$$

`(S.4)` : Set $n \leftarrow n+1$; go to `(S.1)`

- The NE $\mathbf{p}^{\star}(\boldsymbol{\lambda}^{(n)})$ of $\mathcal{G}_{\boldsymbol{\lambda}^{(n)}}$ can be computed using the asynchronous IWFA (convergence is guaranteed under $\boldsymbol{\Upsilon} \succ \mathbf{0}$).

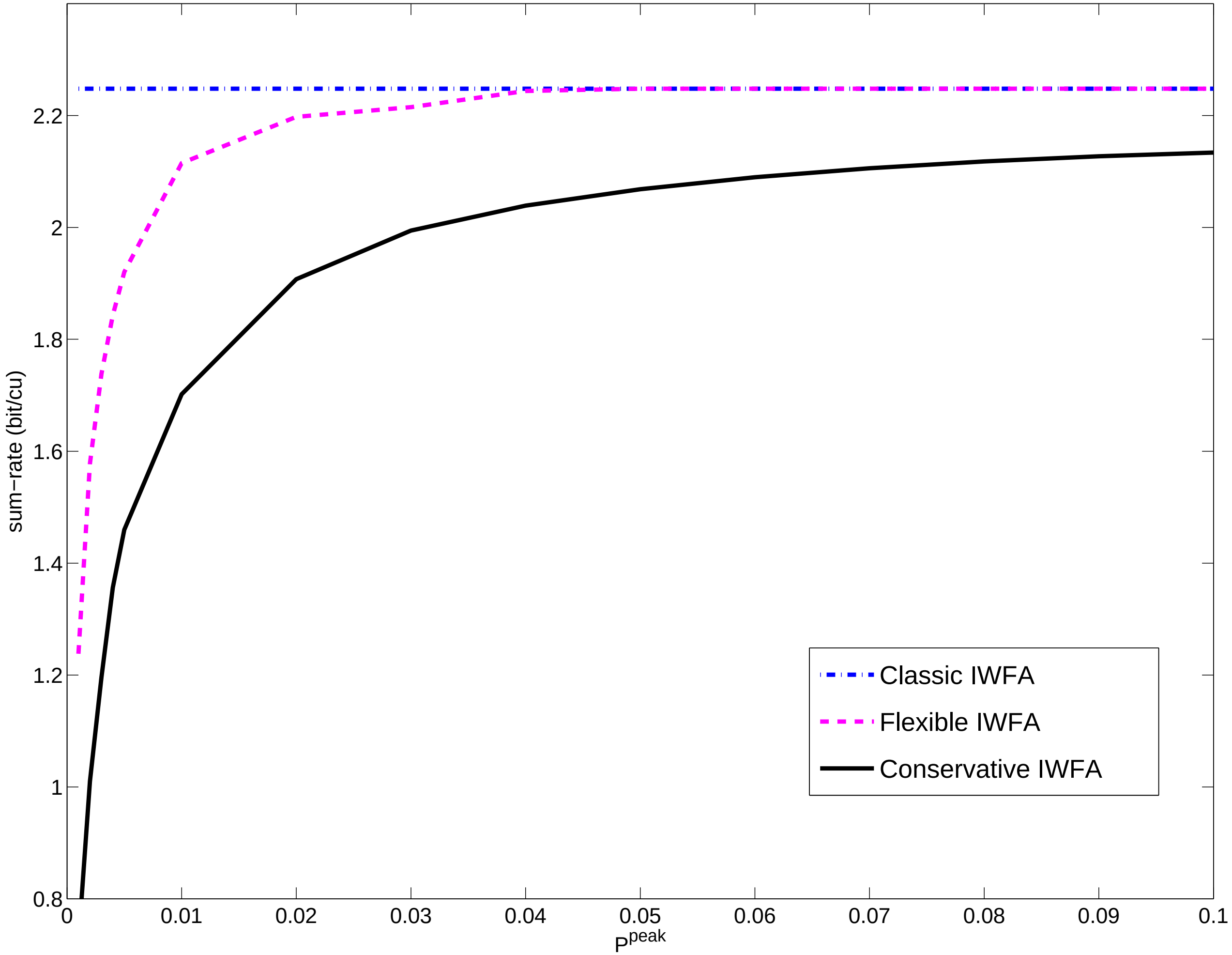
- Distributed implementation: at the interation n, the PUs measure the interference violation $\boldsymbol{\Phi}\left(\boldsymbol{\lambda}^{(n)}\right)$, update the prices $\boldsymbol{\lambda}^{(n+1)}$ via the projection (2), and broadcast $\boldsymbol{\lambda}^{(n+1)}$ to the SUs who play the game $\mathcal{G}_{\boldsymbol{\lambda}^{(n+1)}}$ (asynchronous IWFA).

- **Theorem 3 (Global convergence):** *Suppose* $\boldsymbol{\Upsilon} \succ \mathbf{0}$. *If the step-size* τ *is sufficiently small, then the sequence* $\{\boldsymbol{\lambda}^{(n)}\}_{n=0}^{\infty}$ *generated by Algorithm 1 converges to a solution of the* NCP($\boldsymbol{\Phi}$).

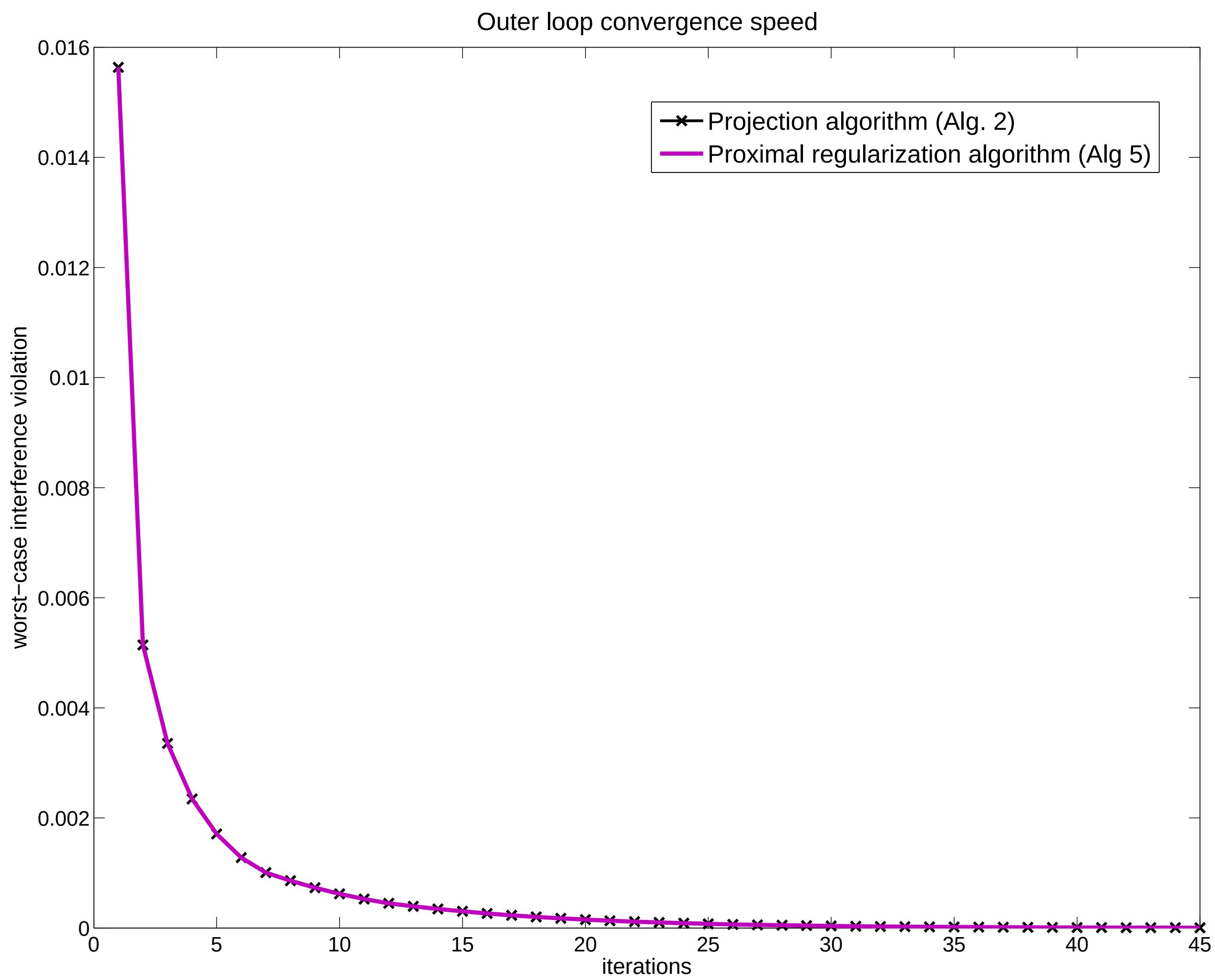
- Several other algorithms have been considered that differ in the trade-off between SUs/PUs signaling, computational complexity, convergence conditions.

Numerical Results

Achieved Sum-Rate

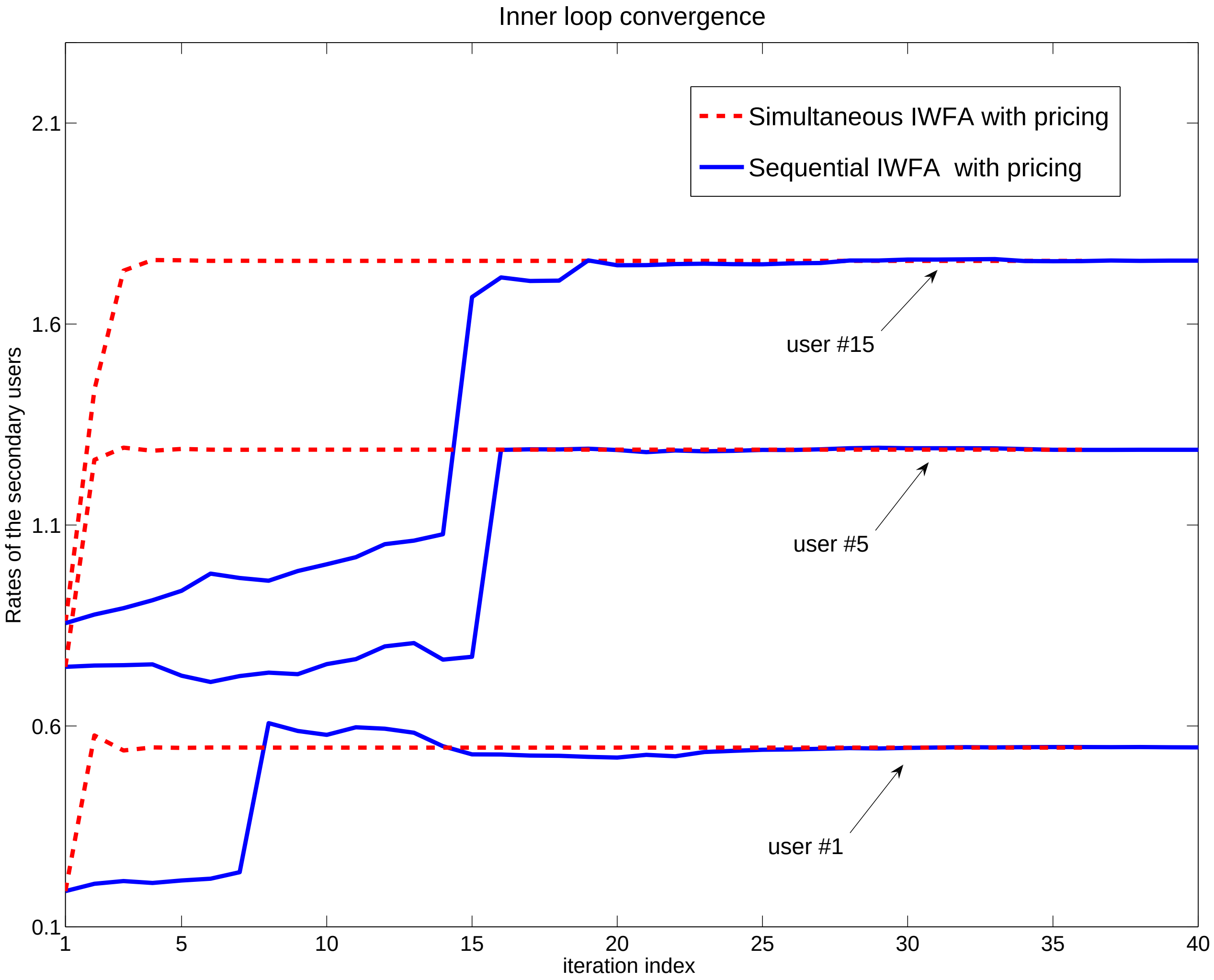


Convergence of Outer Loop



(a)

Convergence of Inner Loop



(b)

Summary

- We have considered different game formulations of relevant wireless networks, starting from simple ad-hoc networks and building up to more complicated cognitive radio systems with temperature-interference constraints.

- The proper way to deal with temperature-interference constraints involves a coupling constraint in the game.

- Variational Inequality theory is a perfect mathematical framework for the analysis and design of such coupled games, both in theory and practice.

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End of Talk

Thank you !!

For more information visit:

http://www.ece.ust.hk/~palomar

Deleted Parts

- The $Q \times Q$ Z-matrix $\boldsymbol{\Upsilon}$

$$[\,\boldsymbol{\Upsilon}\,]_{qr} \triangleq \begin{cases} 1 & \text{if } q = r \\ -\max\limits_{1 \leq k \leq N} \left\{ \dfrac{\left|H_{qr}(k)\right|^2}{\left|H_{rr}(k)\right|^2} \cdot \mathsf{innr}_{qr}(k) \right\} & \text{if } q \neq r, \end{cases} \tag{3}$$

with

$$\mathsf{innr}_{qr}(k) \triangleq \frac{\sigma_r^2(k) + \sum_{r'} |H_{rr'}(k)|^2 p_{r'}^{\max}(k)}{\sigma_q^2(k)}$$