Towards a Scalable Control Theory

Can we find distributed controllers by distributed computation?
Outline

• Positive and Convex-Monotone Systems
  ○ Voltage Stability
  ○ HIV and Cancer Treatment
Positive systems

A linear system is called positive if the state and output remain nonnegative as long as the initial state and the inputs are nonnegative:

\[
\frac{dx}{dt} = Ax + Bu \quad y = Cx
\]

Equivalently, \( A, B \) and \( C \) have nonnegative coefficients except for the diagonal of \( A \).

Examples:

- Probabilistic models.
- Economic systems.
- Chemical reactions.
- Traffic Networks.
Positive Systems and Nonnegative Matrices

Classics:
Mathematics: Perron (1907) and Frobenius (1912)
Economics: Leontief (1936)

Books:
Nonnegative matrices: Berman and Plemmons (1979)
Dynamical Systems: Luenberger (1979)

Recent control related work:
Synthesis by linear programming: Rami and Tadeo (2007)
Switched systems: Liu (2009), Fornasini and Valcher (2010)
Distributed control: Tanaka and Langbort (2010)
Robust control: Briat (2013)

Anders Rantzer, LCCC Linnaeus center  Scalable Control of Convex-Monotone Systems
Suppose the matrix $A$ has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There exists a vector $\xi > 0$ such that $A\xi < 0$. (The vector inequalities are elementwise.)

(iii) There exists a vector $z > 0$ such that $A^Tz < 0$.

(iv) There is a diagonal matrix $P > 0$ such that $A^TP + PA < 0$. 

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Solving the three alternative inequalities gives three different Lyapunov functions:

\[ A \xi < 0 \]
\[ A^T P + PA < 0 \]
\[ A^T z < 0 \]

\[ V(x) = \max_k \left( x_k / \xi_k \right) \]
\[ V(x) = x^T Px \]
\[ V(x) = z^T x \]
Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$
\begin{bmatrix}
 a_{11} & a_{12} & 0 & a_{14} \\
 a_{21} & a_{22} & a_{23} & 0 \\
 0 & a_{32} & a_{33} & a_{32} \\
 a_{41} & 0 & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
 \xi_1 \\
 \xi_2 \\
 \xi_3 \\
 \xi_4
\end{bmatrix}
<
\begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0
\end{bmatrix}
$$

The first node verifies the inequality of the first row.
The second node verifies the inequality of the second row.

... 

*Verification is scalable!*
A Distributed Search for Stabilizing Gains

Suppose
\[
\begin{bmatrix}
    a_{11} - \ell_1 & a_{12} & 0 & a_{14} \\
    a_{21} + \ell_1 & a_{22} - \ell_2 & a_{23} & 0 \\
    0 & a_{32} + \ell_2 & a_{33} & a_{32} \\
    a_{41} & 0 & a_{43} & a_{44}
\end{bmatrix}
\geq 0 \text{ for } \ell_1, \ell_2 \in [0, 1].
\]

For stabilizing gains \( \ell_1, \ell_2 \), find \( 0 < \mu_k < \xi_k \) such that
\[
\begin{bmatrix}
    a_{11} & a_{12} & 0 & a_{14} \\
    a_{21} & a_{22} & a_{23} & 0 \\
    0 & a_{32} & a_{33} & a_{32} \\
    a_{41} & 0 & a_{43} & a_{44}
\end{bmatrix}
\begin{bmatrix}
    \xi_1 \\
    \xi_2 \\
    \xi_3 \\
    \xi_4
\end{bmatrix}
+ \begin{bmatrix}
    -1 & 0 \\
    1 & -1 \\
    0 & 1 \\
    0 & 0
\end{bmatrix}
\begin{bmatrix}
    \mu_1 \\
    \mu_2
\end{bmatrix}
< \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    0
\end{bmatrix}
\]

and set \( \ell_1 = \mu_1 / \xi_1 \) and \( \ell_2 = \mu_2 / \xi_2 \). Every row gives a local test. Distributed synthesis by linear programming (gradient search).
Examples: Transportation Networks

- Cloud computing / server farms
- Heating and ventilation in buildings
- Traffic flow dynamics
- Production planning and logistics
**Externally Positive Systems**

$G \in RH_{\infty}^{m \times n}$ is called *externally positive* if the corresponding impulse response $g(t)$ is nonnegative for all $t$. The set of all such matrices is denoted $PH_{\infty}^{m \times n}$.

Suppose $G, H \in PH_{\infty}^{n \times n}$. Then

- $GH \in PH_{\infty}^{n \times n}$
- $aG + bH \in PH_{\infty}^{n \times n}$ when $a, b \in \mathbb{R}_+$.
- $\|G\|_{\infty} = \|G(0)\|$.
- $(I - G)^{-1} \in PH_{\infty}^{n \times n}$ if and only if $G(0)$ is Schur.

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Scalable Control of Convex-Monotone Systems
\( \mathbf{G} \in \mathbb{RH}_{\infty}^{m \times n} \) is called \textit{positively dominated} if \( |G_{jk}(i\omega)| \leq G_{jk}(0) \) for \( \omega \in \mathbb{R} \). The set of all such matrices is denoted \( \mathbb{DH}_{\infty}^{m \times n} \).

Suppose \( \mathbf{G}, \mathbf{H} \in \mathbb{DH}_{\infty}^{n \times n} \). Then

- \( \mathbf{GH} \in \mathbb{DH}_{\infty}^{n \times n} \)
- \( a\mathbf{G} + b\mathbf{H} \in \mathbb{DH}_{\infty}^{n \times n} \) when \( a, b \in \mathbb{R}_+ \).
- \( \|\mathbf{G}\|_{\infty} = \|\mathbf{G}(0)\| \).
- \( (I - \mathbf{G})^{-1} \in \mathbb{DH}_{\infty}^{n \times n} \) if and only if \( \mathbf{G}(0) \) is Schur.
Example 3: Mass-spring system

\[
\ddot{x}_i + d_i \dot{x}_i + k_i x_i = \sum_j \ell_{ij} (x_j - x_i) + w_i
\]

\[
(s^2 + d_i s + k_i + \sum_j \bar{\ell}_{ij}) X_i(s) = \sum_j \left( \ell_{ij} X_j(s) + (\ell_{ij} - \ell_{ij}) X_i(s) \right) + W_i(s)
\]

\[
X = (A + ELF)X + BW
\]

The transfer matrices \(B\), \(E\) and \(A + ELF\) are positively dominated for all \(L \in D\) provided that \(d_i \geq k_i + \sum_j \bar{\ell}_{ij}\).
Max-separable Lyapunov Functions

Max-separable: \( V(x) = \max\{V_1(x_1), \ldots, V_n(x_n)\} \)

**Theorem.** Let \( \dot{x} = f(x) \) be a monotone system such that the origin globally asymptotically stable and the compact set \( X \subset \mathbb{R}^n_+ \) is invariant. Then there exist strictly increasing functions \( V_k : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) for \( k = 1, \ldots, n \), such that \( V(x) = \max\{V_1(x_1), \ldots, V_n(x_n)\} \) satisfies

\[
\frac{d}{dt} V(x(t)) = -V(x(t))
\]

along all trajectories in \( X \).

[Rantzer, Rüffer, Dirr, CDC-13]
Proof idea

$\begin{align*}
  t &= 0 \\
  t &= 1 \\
  t &= 2 \\
  t &= 3
\end{align*}$
The system

\[ \dot{x}(t) = f(x(t), u(t)), \quad x(0) = a \]

is a monotone system if its linearization is a positive system. It is a convex monotone system if every row of \( f \) is also convex.

**Theorem.** [Rantzer/ Bernhardsson (2014)]

For a convex monotone system \( \dot{x} = f(x, u) \), each component of the trajectory \( \phi_t(a, u) \) is a convex function of \((a, u)\).
Outline

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- Voltage Stability
- HIV and Cancer Treatment
The power \( p = iu_2 \) delivered to the load is upper bounded by

\[
p = i(u_1 - Ri) \leq \frac{u_1^2}{4R}.
\]

An active load:

\[
\frac{di}{dt} = \frac{\hat{p}}{u_1 - Ri} - i.
\]

where \( \hat{p} \) is the power demand.

Voltage collapse occurs if \( \hat{p} \) is too large!
Node 3 is an active load with

\[ \frac{di_3}{dt} = \frac{\hat{p}(y_1 + y_2)}{y_1u_1 + y_2u_2 - i_3} - i_3 \]

For constant generator voltages \( u_1 \) and \( u_2 \), the load voltage \( u_3 = y_1u_1 + y_2u_2 - i_3 \) could shrink to zero in finite time, which means voltage collapse.
Voltages at generators $u^G$ and loads $u^L$ are mapped into external currents $i^G$ and $i^L$ according to

$$
\begin{bmatrix}
-i^G(t) \\
i^L(t)
\end{bmatrix} =
\begin{bmatrix}
Y^{GG} & Y^{GL} \\
Y^{LG} & Y^{LL}
\end{bmatrix}
\begin{bmatrix}
u^G(t) \\
u^L(t)
\end{bmatrix}
$$

The load model: \( \frac{di^L}{dt}(t) = \frac{\hat{p}_k}{u_k^L(t)} - i^L_k(t) \) gives

$$
\frac{di^L}{dt}(t) = \frac{\hat{p}}{([Y^{LL}]^{-1}(i^L - Y^{LG}u^G))} - i^L(t)
$$

*This system is convex-monotone with state $i^L$ and input $-u^G$, so

$$
i^G, -u^L, i^L, \frac{di^L}{dt} \text{ and } \frac{di^G}{dt}
$$

are all convex functions of $u^G$
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Combination Therapy is a Control Problem

Evolutionary dynamics:

\[ \dot{x} = \left( A - \sum_{i} u_i D^i \right) x \]

Each state \( x_k \) is the concentration of a mutant. (There can be hundreds!) Each input \( u_i \) is a drug dosage.

\( A \) describes the mutation dynamics without drugs, while \( D^1, \ldots, D^m \) are diagonal matrices modeling drug effects.

Determine \( u_1, \ldots, u_m \geq 0 \) with \( u_1 + \cdots + u_m \leq 1 \) such that \( x \) decays as fast as possible!

[Jonsson, Rantzer, Murray, ACC 2014]
Optimizing Decay Rate

Stability of the matrix $A - \sum_i u_i D^i + \gamma I$ is equivalent to existence of $\xi > 0$ with

$$(A - \sum_i u_i D^i + \gamma I)\xi < 0$$

For row $k$, this means

$$A_k\xi - \sum_i u_i D_k^i \xi_k + \gamma \xi_k < 0$$

or equivalently

$$\frac{A_k\xi}{\xi_k} - \sum_i u_i D_k^i + \gamma < 0$$

Maximizing $\gamma$ is convex optimization in $(\log \xi_i, u_i, \gamma)$!
Evolutionary dynamics:

\[ \dot{x}(t) = \left( A - \sum_{i} u_i(t) D_i \right) x(t) \]

Can we get faster decay using time-varying \( u(t) \) based on measurements of \( x(t) \)?
The evolutionary dynamics can be written as a convex monotone system:

\[
\frac{d}{dt} \log x_k(t) = \frac{A_k x(t)}{x_k(t)} - \sum_i u_i(t) D^i_k
\]

Hence the decay of \( \log x_k \) is a convex function of the input and optimal trajectories can be found even for large systems.
Example

\[
A = \begin{bmatrix}
-\delta & \mu & \mu & 0 \\
\mu & -\delta & 0 & \mu \\
\mu & 0 & -\delta & \mu \\
0 & \mu & \mu & -\delta
\end{bmatrix}
\]

clearance rate \( \delta = 0.24 \text{ day}^{-1} \), mutation rate \( \mu = 10^{-4} \text{ day}^{-1} \)
and replication rates for viral variants and therapies as follows

<table>
<thead>
<tr>
<th>Virus variant</th>
<th>Therapy 1</th>
<th>Therapy 2</th>
<th>Therapy 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type 1 ((x_1))</td>
<td>(D^1_1 = 0.05)</td>
<td>(D^2_1 = 0.10)</td>
<td>(D^3_1 = 0.30)</td>
</tr>
<tr>
<td>Type 2 ((x_2))</td>
<td>(D^1_2 = 0.25)</td>
<td>(D^2_2 = 0.05)</td>
<td>(D^3_2 = 0.30)</td>
</tr>
<tr>
<td>Type 3 ((x_3))</td>
<td>(D^1_3 = 0.10)</td>
<td>(D^2_3 = 0.30)</td>
<td>(D^3_3 = 0.30)</td>
</tr>
<tr>
<td>Type 4 ((x_4))</td>
<td>(D^1_4 = 0.30)</td>
<td>(D^2_4 = 0.30)</td>
<td>(D^3_4 = 0.15)</td>
</tr>
</tbody>
</table>
Example

Optimized drug doses:

Total virus population:
Summary

- Scalability for Positive and Convex-Monotone Systems
- Voltage Stability
- HIV and Cancer Treatment