Synchronization of oscillators: Feasibility and Non-local analysis

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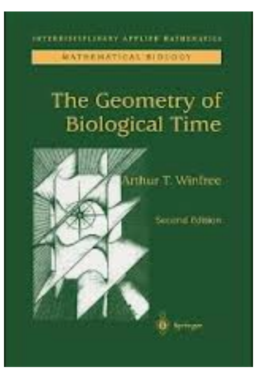
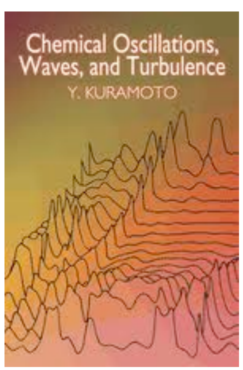
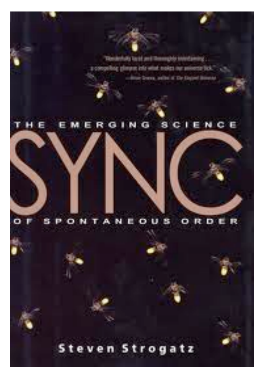
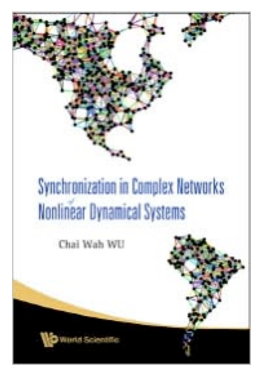
A Brief History of Sync

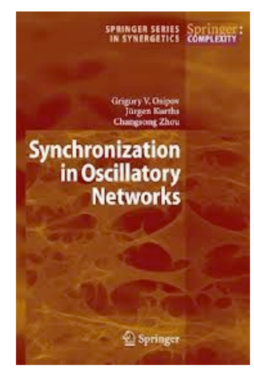
- Sync experiments with coupled pendulum clocks [C. Huygens, 1665]
- Sync in mathematical biology [A. Winfree '80, S.H. Strogatz '03, . . .]
- Sync in physics and chemistry [Y. Kuramoto '83, M. Mézard et al. '87. . .]
- Sync in neural networks [F.C. Hoppensteadt and E.M. Izhikevich '00, . . .]
- Sync in complex networks [C.W. Wu '07, S. Bocaletti '08, . . .]
- . . . and numerous technological applications [F. Dörfler and F. Bullo, '14]

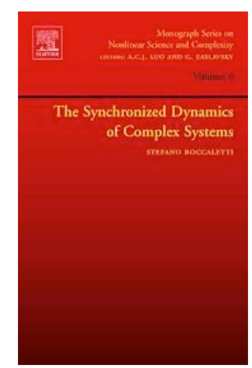






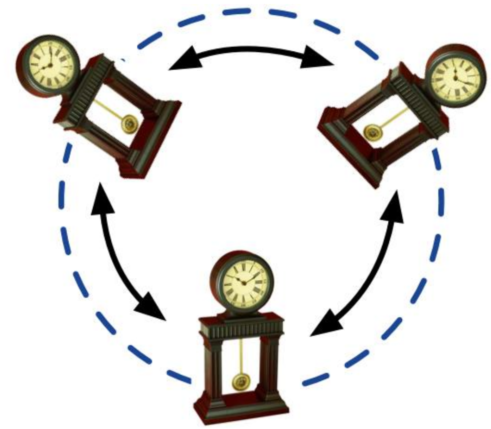






Coupled Phase Oscillators

$\exists$ various models of oscillators & interactions

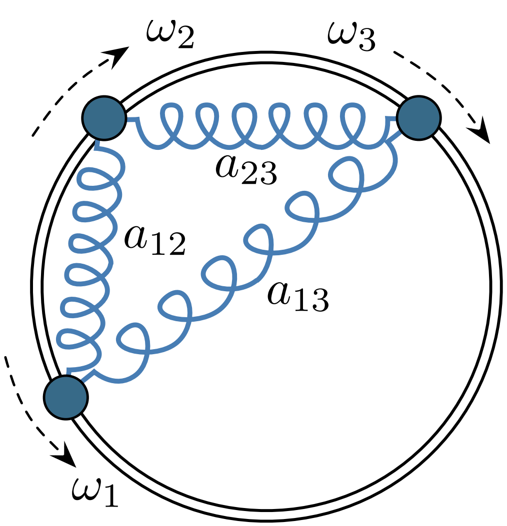


Today: canonical coupled oscillator model
[A. Winfree '67, Y. Kuramoto '75]

Coupled Kuramoto oscillator model:

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

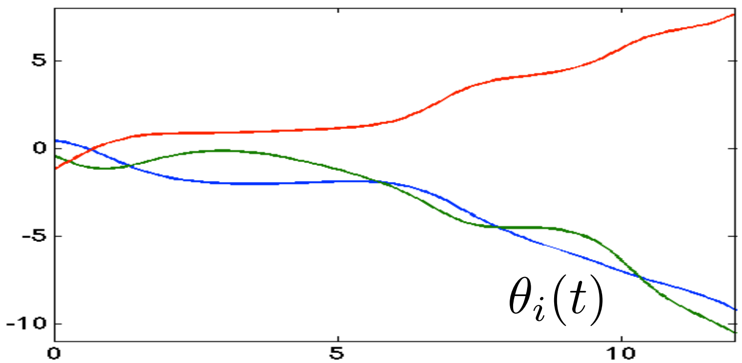
- n **oscillators** with phase $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- elastic **coupling** with strength $a_{ij} = a_{ji}$
- undirected & connected **graph** $G = (\mathcal{V}, \mathcal{E}, A)$



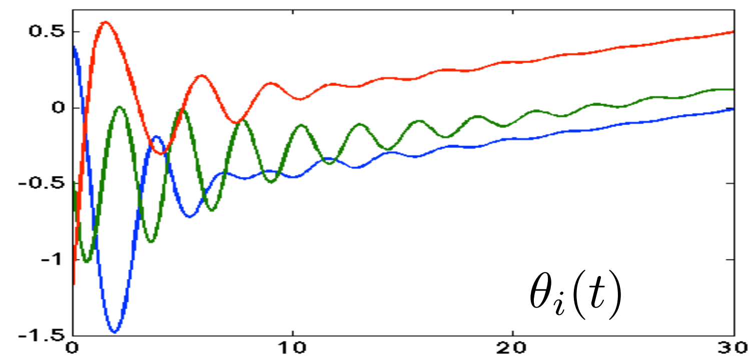
Problem I: Existence of Synchronization?

Synchronization is a **trade-off:**
coupling vs. heterogeneity

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



coupling small & $|\omega_i - \omega_j|$ large
$\Rightarrow$ incoherence & no sync



coupling large & $|\omega_i - \omega_j|$ small
$\Rightarrow$ coherence & frequency sync

Existence sync trajectory $\Leftrightarrow$ solution to nonlinear system of equations

Some central questions:
(still after 45 years of work)

- quantify "coupling" vs. "heterogeneity"
- necessary & sufficient conditions

Problem II: Region of attraction?

$$\dot\theta_i = \omega_i - \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j)$$

If frequency synchronized trajectory $\theta_i(t) = \theta_i^0 + \omega_{sync}\, t \qquad \forall i$

How large is the ***basin of attraction***?

Symmetry ➔ $\omega_{sinc} = \frac{1}{n}\sum_{j=1}^n \omega_i$ assumed = 0 w.l.o.g.

Frequency synchronized trajectory = equilibrium $\dot\theta_i = 0$

Many results available, almost all* ***avoid complexity of circle manifold*** by

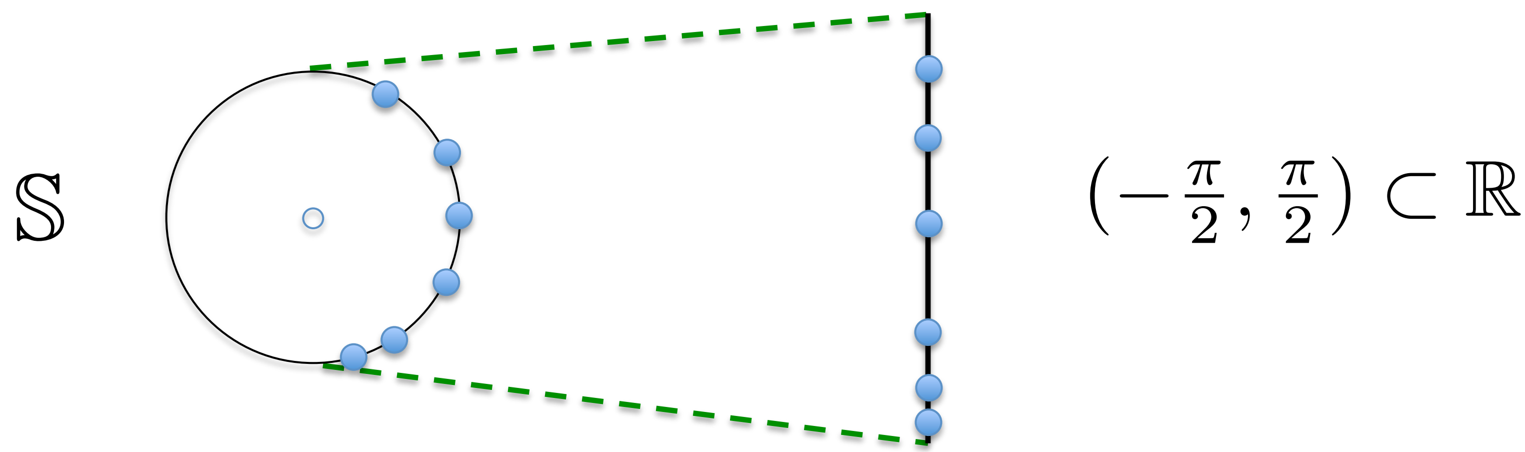
- Assuming phases initially in a « convenient » set,
- Ensuring they remain in the set during synchronization.

* Notable exception *[Sepulchre Paley Leonard TaC 07]*

Problem II: Region of attraction?

Convenient set 1: Semi-circle

$$\mathcal{C} := \{\theta \in \mathbb{S}^n : \exists \delta : \theta_i \in [-\tfrac{\pi}{2} + \delta, \tfrac{\pi}{2} + \delta], \forall i\}$$



Convenient set 2: small differences between neighbors (< π/2)

$$\mathcal{B} := \{\theta \in \mathbb{S}^n : (i,j) \in E \Rightarrow |\theta_i - \theta_j| < \pi/2\}$$

Implies « classical » symmetric consensus on frequencies

$$\frac{d}{dt}\dot{\theta}_i = \sum_{j=1}^{n} \tilde{a}_{ij}(\theta)(\dot{\theta}_j - \dot{\theta}_i) \qquad \text{with} \qquad \tilde{a}_{ij}(\theta) = \tilde{a}_{ji}(\theta) \geq 0$$

Problem II: Region of attraction?

Question: For nontrivial classes of weights and natural frequencies, existence of a frequency synchronized trajectory that

- Strong version: is ***almost globally*** attractive
- Weak version: has a ***large basin of attraction:*** not included in

$$\mathcal{B} := \{\theta \in \mathbb{S}^n : (i,j) \in E \Rightarrow |\theta_i - \theta_j| < \pi/2\}$$
$$\mathcal{C} := \{\theta \in \mathbb{S}^n : \exists \delta : \theta_i \in [-\tfrac{\pi}{2} + \delta, \tfrac{\pi}{2} + \delta], \forall i\}$$

Specific conjecture:

In the following cases, every ***locally asymptotically stable*** synchronized trajectory is ***almost globally attractive***

1) Acyclic topologies: interaction graph is a tree
2) Fully connected topology with equal weights: $a_{ij} = \frac{K}{n}, \quad \forall i,j$