Majority Consensus by Local Polling

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Lund Workshop on Dynamics and Control in Networks
October 2014
1785, Marquis de Condorcet’s weak law of large numbers

- in a large population of voters, and each one independently votes correctly with probability $\alpha > 1/2$
- as population size grows, probability that the outcome of a majority vote is correct converges to one

Information is efficiently aggregated
Aggregating in a network

Majority Consensus by Local Polling
Binary majority consensus

Desired outcome and metrics
- Nodes end with opinion held by majority of nodes
- Node can probe neighbours and update opinion accordingly using little (constant) memory
- Probability of error (convergence to incorrect consensus)
- Time to convergence

Applications
- Occurrence of a given event in cooperative decision making
- Voting in distributed systems
- Routine to solve more elaborate distributed decision making instances
$G = (V, E)$ simple connected graph on $|V| = n$ vertices

Each vertex either red (1) or blue (0).
Initial proportion of blues is $\alpha \in (1/2, 1)$

**GOAL:** Local algorithm for inferring the majority state.

- Does the graph settle into one colour?
- If so, how does the graph structure and the initial distribution affect which colour wins?
- How long does it take?
- Distributed consensus [known results]
- Interval consensus [Draief, Vojnovic ’12]
- Local polling [Abdullah, Draief ’14]
Continuous-time Interaction Model

- Connected undirected graph $G = (V, E)$, $|V| = n$
- $\alpha n$ nodes hold 0 and $(1 - \alpha)n$ nodes hold 1, $\alpha \in (1/2, 1)$
- Nodes $i$ and $j$ interact at rate $q_{ij} = q_{ji}$, $q_{ij} \neq 0$ iff $(i, j) \in E$

Markov chain

- $(X_t)_{t \geq 0}$ continuous-time Markov chain with rate matrix $Q$, $q_{ii} = -\sum_{i \neq j} q_{ij}$
- $(\pi_i)_{i \in V}$ stationary distribution is uniform on $V$. Mixing time:

$$|\mathbb{P}_j(X_t = i) - 1/n| = O\left(e^{-\lambda_2(Q)t}\right)$$

where $\lambda_2(Q) = \inf\{\sum_{i,j} q_{ij}(x_i - x_j)^2/2, ||x|| = 1, x^T1 = 0\}$
Performance of voter model

Node $i$ contacts $j$ at rate $q_{ij}$ and $i$ updates to $j$'s state

**Theorem [Hassin-Peleg ’01]**

- The number of nodes in state 1 is a martingale.
- Probability of reaching (wrong) consensus at 1 is $1 - \alpha$.
- Time to convergence of voter model $O(n/(\lambda_2(Q)))$. 
Complete graph

- Each edge has rate $1/(n - 1)$. Number of agents with opinion 1 evolves as Birth-Death process

$$\lambda_{k,k+1} = \lambda_{k,k-1} = \frac{k(n-k)}{n-1}.$$ 

- Time to convergence $= O(n)$
General graphs

- Conductance $\eta(Q) = \inf_{A \subseteq V} \frac{\sum_{i \in A, j \in A^c} q_{ij}}{|A||A^c|/n}$

- Markov chain tracking the number of nodes in state 0 evolves at least $\eta(Q)$ times as fast as on the complete graph, since

$$\sum_{i \in A, j \in A^c} q_{ij} \geq \eta(Q) \frac{|A||A^c|}{n}$$

- Time to convergence $O(n/\eta(Q))$, 

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Cheeger's inequality

- Conductance: $\eta(Q) = \inf_{A \subset V} \frac{\sum_{i \in A, j \in A^c} q_{ij}}{|A||A^c|/n}$
- Spectral Gap: $\lambda_2(Q) = \inf\{\sum_{i,j} q_{ij}(x_i - x_i)^2/2, \|x\| = 1, x^T1 = 0\}$

$$2\lambda_2(Q) \leq \eta(Q).$$

- Time to convergence of voter model $O(n/(\lambda_2(Q)))$.

Let $S$ of size $k$ be the subset realising the inf in $\eta(Q)$ and let $x$ such that $x_i = -\sqrt{\frac{n-k}{kn}}, i \in S$ and $x_i = \sqrt{\frac{k}{(n-k)n}}, i \in S^c$. 

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At each interaction of \((i, j)\) occurring at rate \(q_{ij}\)

\[
x_i(t) = x_j(t) = \frac{x_i(t-) + x_j(t-)}{2}.
\]

Theorem [Boyd et al. ’06, Aldous ’12]

- Algorithm converges to the average value, using \(O(Poly(\log(n)))\) memory per node.
- Time to convergence to up \(O(1/n)\) error of the average is \(\Theta(\log(n)/\lambda_2(Q))\)
Distributed averaging: Proof

Let $R(t) = \|x(t)\|^2$. When an $i, j$ interaction takes place $R(t)$ reduces by $(x_i - x_j)^2/2$.

$$
\mathbb{E}(dR(t) \mid x(t) = x) = \sum_{i,j} q_{ij} \left(2 \left(\frac{x_i + x_j}{2}\right)^2 - (x_i^2 + x_j^2)\right)
$$

$$
= - \sum_{i,j} q_{ij} \frac{(x_i - x_j)^2}{2} dt
$$

(Assume that $\sum_i x_i(0) = 0$)

$$
\leq - \lambda_2(Q) \|x\|^2 dt
$$

In particular

$$
\mathbb{E}\|x(t)\|^2 \leq \|x(0)\|^2 e^{-\lambda_2(Q)t}
$$
Could we use less memory and still guarantee small error?

Theorem: Impossibility

- Connected undirected graph $G = (V, E)$, $|V| = n$,
- $\alpha n$ nodes in 0 and $(1 - \alpha)n$ nodes in 1, $\alpha \in (1/2, 1)$,
  $2\alpha - 1$ is the voting margin.

No 1-bit distributed algorithm can solve the majority consensus problem.

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Land, Belew, “No perfect two-state cellular automata for density classification exists”, PRL 74, 5148-5150, 1995
Ternary Consensus

- $\alpha n$ nodes hold 0 and $(1 - \alpha) n$ nodes hold 1,
- Additional state $e$ for undecided nodes, $q_{i,j} = 1/n$, $\forall i, j$

**Theorem [PVV ’09]**

Probability of reaching wrong consensus $1$. For $n$ large,

$$P_{\text{error}} = (1 + o(1))2^{-D(\alpha||\frac{1}{2})n}$$

where $D(\alpha||\frac{1}{2})$ is the Kullback-Leibler divergence. $T$ time to convergence, $\mathbb{E}(T) = (1 + o(1)) \log n$.

- Results (seem to) hold for expander but fail for the line.
- Generalises beyond binary consensus [Babaee, Draief ’14]
Binary Consensus with two undecided states

Averaging-like updates: States $0 < e_0 < e_1 < 1$.
Rules: Swaps + Annihilation

Kashyap, Basar, Srikant, “Quantized consensus” Automatica, 1192-1203, 2007
Bénédit, Thiran, Vetterli, Interval consensus: From quantized gossip to voting, ICASSP 2009
Mean-field analysis (Complete graph)

Let $q_{ij} = \frac{1}{n-1}$, $i \neq j$ and $X(t) = (|S_0(t)|, |S_{e_0}(t)|, |S_{e_1}(t)|, |S_1(t)|)$ is a Markov process with the following transition rates

$$
\begin{align*}
\rightarrow & \quad \left\{ \begin{array}{cc}
(|S_0(t)| - 1, |S_{e_0}(t)| + 1, |S_{e_1}(t)| + 1, |S_1(t)| - 1) : & \frac{|S_0(t)||S_1(t)|}{n-1} \\
(|S_0(t)|, |S_{e_0}(t)| - 1, |S_{e_1}(t)| + 1, |S_1(t)|) : & \frac{|S_{e_0}(t)||S_1(t)|}{n-1} \\
(|S_0(t)|, |S_{e_0}(t)| + 1, |S_{e_1}(t)| - 1, |S_1(t)|) : & \frac{|S_0(t)||S_{e_1}(t)|}{n-1}.
\end{array} \right.
\end{align*}
$$
By Kurtz’s theorem, \( X(t)/n \) converges to 
\( (s_0(t), s_{e_0}(t), s_{e_1}(t), s_1(t)) \)

\[
\begin{align*}
    s'_0(t) &= -s_1(t)s_0(t) \\
    s'_1(t) &= -s_0(t)s_1(t) \\
    s'_{e_1}(t) &= s_1(t)(1 - s_1(t)) - (s_0(t) + s_1(t))s_{e_1}(t)
\end{align*}
\]

with \( s_{e_0}(t) = 1 - s_0(t) - s_{e_1}(t) - s_1(t), \ t \geq 0. \)
Proposition [Draief, Vojnovic ’10]

For large $t$,

$$s_{e_1}(t) \sim (2\alpha - 1) \frac{1 - \alpha}{\alpha} te^{-(2\alpha-1)t}$$

$$s_1(t) \sim (2\alpha - 1) \frac{1 - \alpha}{\alpha} e^{-(2\alpha-1)t}.$$

In particular, $t_{n,\alpha}^1$ and $t_{n,\alpha}^{e_1}$ times nodes in 1 and $e_1$ to disappear

$$t_{n,\alpha}^1 = \frac{1}{2\alpha - 1} \log(n) + O(1)$$

$$t_{n,\alpha}^{e_1} = \frac{1}{2\alpha - 1} \left[\log(n) + \log(\log(n))\right] + O(1).$$
Minority states

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Theorem [Draief, Vojnovic ’12]

Let $T$ be the time until there are only nodes in state 0 and $e_0$.

$$\mathbb{E}(T) = \Theta(\log n / \delta(Q, \alpha))$$

where $\delta(Q, \alpha) = \min_{S \subseteq V, |S| = (2\alpha - 1)n} \min_{\lambda \in \text{Spec}(Q_S)} |\lambda|$

$$Q_S = \begin{bmatrix}
\text{diag}(q_{ii}, i \in S) & 0 \\
(q_{ij})_{i \in S^c, j \in S} & (q_{ij})_{i, j \in S^c}
\end{bmatrix}$$
First phase: \( Z_i(t) (A_i(t)) \) indicator that \( i \) in state 0 (1) at \( t \)

\[
(Z, A) \rightarrow \begin{cases} 
(Z - e_i, A - e_j) & : q_{i,j} Z_i A_j \\
(Z - e_i + e_j, A) & : q_{i,j} Z_i (1 - A_j - Z_j) \\
(Z, A - e_i + e_j) & : q_{i,j} A_i (1 - A_j - Z_j)
\end{cases}
\]

Second phase: \( B_i(t) \) indicator that node \( i \) is in state \( e_1 \) at \( t \)

\[
(Z, B) \rightarrow \begin{cases} 
(Z - e_i + e_j, B - e_j) & : q_{i,j} Z_i B_j \\
(Z - e_i + e_j, B) & : q_{i,j} Z_i (1 - B_j - Z_j) \\
(Z, B - e_i + e_j) & : q_{i,j} B_i (1 - B_j - Z_j)
\end{cases}
\]
Piecewise-linear dynamical system

\[
\frac{d}{dt} \mathbb{E}(Y_i(t)) = -\left(\sum_{l \in V} q_{i,l}\right) \mathbb{E}(Y_i(t)) + \sum_{j \in V} q_{i,j} \mathbb{E}(Y_j(t)(1 - Z_i(t))).
\]

Dynamics reduces to \( Y(t) = (Y_i(t))_{i \in V}, \)

\[
\frac{d}{dt} \mathbb{E}_k(Y(t)) = Q_{S_k} \mathbb{E}_k(Y(t)),
\]

for \( t \in [t_k, t_{k+1}) \) during which \( \{S_0(t) = S_k\} \) and \( Q_{S_k} \) is given by

\[
Q_S(i,j) = \begin{cases} 
-\sum_{l \in V} q_{i,l}, & i = j \\
q_{i,j}, & i \notin S, j \neq i \\
0, & i \in S, j \neq i.
\end{cases}
\]
Solution

Proposition

\[ \mathbb{E}(Y(t)) = \mathbb{E} \left[ e^{\lambda(t)} Y(0) \right] \]

where \( \lambda(t) = Q_{S_k}(t - t_k) + \sum_{i=0}^{k-1} Q_{S_i}(t_{i+1} - t_i) \).

Lemma

For any finite graph \( G \), there exists \( \delta(Q, \alpha) > 0 \) such that, for any non-empty subset of vertices \( S \) with \( |S| \in [(2\alpha - 1)n, \alpha n] \), if \( \lambda \) is an eigenvalue of the matrix \( Q_S \), then

\[ \lambda \leq -\delta(G, \alpha) < 0. \]
Proof: Spectrum of $Q_S$

$$Q_S = \begin{bmatrix} \text{diag}(q_{ii}, \ i \in S) & 0 \\ (q_{ij})_{i \in S^c, \ j \in S} & (q_{ij})_{i, j \in S^c} \end{bmatrix}$$

- First $\left( q_{ii} = -\sum_{l \neq i} q_{i,l} \right)$, $i \in S$ are eigenvalues of $Q_S$
- The remaining eigenvalues correspond to eigenvectors $x = (0, \ldots, 0, x_S^T, x_{S^c})$. Let $W \subset S^c$, for $i \in W$, $x_i \neq 0$

$$-\lambda = x^T Q_S x$$

$$= \sum_{i \in W} \sum_{j \in S} q_{i,j} x_i^2 + \sum_{i \in W, j \in S^c \setminus W} q_{i,j} x_i^2 + \frac{1}{2} \sum_{i, j \in W} q_{i,j} (x_i - x_j)^2$$
Proof

Note that

$$\mathbb{E}(Y(t)) = \mathbb{E}\left[e^{\lambda(t)} Y(0)\right]$$

where $$\lambda(t) = Q_{S_k}(t - t_k) + \sum_{l=0}^{k-1} Q_{S_l}(t_{l+1} - t_l)$$

By Jensen’s and matrix norm inequalities,

$$\|\mathbb{E}(Y(t))\|_2 \leq \mathbb{E}\left[\|e^{Q_{S_k}(t-t_k)}\| \prod_{l=0}^{k-1} \|e^{Q_{S_l}(t_{l+1}-t_l)}\| \|Y(0)\|_2\right] \leq \sqrt{n} e^{-\delta(G,\alpha)t}$$

Therefore, by Cauchy-Schwartz, we have

$$\mathbb{P}(Y(t) \neq 0) \leq \sum_{i \in V} \mathbb{E}(Y_i(t)) \leq n e^{-\delta(G,\alpha)t}$$

We conclude since $$\mathbb{E}(T_0) = \int_0^\infty \mathbb{P}(Y(t) \neq 0) dt.$$
Complete graph

Corollary

An application of the theorem to complete graph $q_{i,j} = \frac{1}{n-1}$ for all $i \neq j$, yields

$$\mathbb{E}(T) \leq 2 \frac{1}{2\alpha - 1} \log(n).$$

Exact asymptotics

A direct analysis of the dynamics of the 1st phase

$$\mathbb{E}(T_1) = \frac{n - 1}{|S_0| - |S_1|} \left( H_{|S_1|} + H_{|S_0| - |S_1|} - H_{|S_0|} \right)$$

where $H_k = \sum_{i=1}^{k} \frac{1}{i}$

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Various initial conditions

- $|S_0| - |S_n| = (2\alpha - 1)n$, $\alpha$ a constant larger than $1/2$

$$\mathbb{E}(T_1) = \frac{1}{2\alpha - 1} \log(n) + O(1).$$

- If $|S_0| = |S_1|$

$$\mathbb{E}(T_1) = \frac{\pi^2}{6} n(1 + o(1)).$$

- $\mu_n = (|S_0| - |S_1|)/n$ is strictly positive but small ($o(1)$),

$$\mathbb{E}(T_1) = \frac{1}{\mu_n} \log(n\mu_n) + O(1).$$
Complete Graph: Theory v. Simulation

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**Star Network:** \( q_{1,i} = q_{i,1} = \frac{1}{n-1}, \ i \neq 1 \) and \( q_{i,j} = 0, \ i, j \neq 1 \).

\[ \mathbb{E}(T_i) \leq \frac{1}{2\alpha - 1} n \log(n). \]

Using, direct calculation

\[ \mathbb{E}(T_1) = \frac{1}{(2\alpha - 1)(3 - 2\alpha)} n \log(n) + O(n) \]

**ER-graph:** \( q_{i,j} = \frac{1}{np_n} X_{i,j} X_{i,j} \) i.i.d. Bernoulli r.v. with mean \( c \frac{\log(n)}{n} \), \( c > \frac{2}{2\alpha - 1} \), for \( h^{-1} \) the inverse of \( h(x) = x \log(x) + 1 - x \),

\[ \mathbb{E}(T_i) \leq \frac{1}{(2\alpha - 1)h^{-1} \left( \frac{2}{c(2\alpha - 1)} \right)} \log(n) + O(1) \]

**Path:** \( \mathbb{E}(T_i) \leq \frac{16(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1) \)

**Ring:** \( \mathbb{E}(T_i) \leq \frac{4(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1) \).
Majority Consensus by Local Polling
Star Network: $q_{1,i} = q_{i,1} = \frac{1}{n-1}$, $i \neq 1$ and $q_{i,j} = 0$, $i,j \neq 1$.

$\mathbb{E}(T_i) \leq \frac{1}{2^\alpha - 1} n \log(n)$. Using, direct calculation

$$\mathbb{E}(T_1) = \frac{1}{(2\alpha - 1)(3 - 2\alpha)} n \log(n) + O(n)$$

ER-graph: $q_{i,j} = \frac{1}{n \rho_n} X_{i,j} X_{i,j}$ i.i.d. Bernoulli r.v. with mean

$\rho_n = c \frac{\log(n)}{n}$, $c > \frac{2}{2\alpha - 1}$, for $h^{-1}$ the inverse of

$h(x) = x \log(x) + 1 - x$,

$$\mathbb{E}(T_i) \leq \frac{1}{(2\alpha - 1)h^{-1} \left( \frac{2}{c(2\alpha - 1)} \right)} \log(n) + O(1)$$

Path: $\mathbb{E}(T_i) \leq \frac{16(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1)$

Ring: $\mathbb{E}(T_i) \leq \frac{4(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1)$. 

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Path and Ring

- **Star Network:** \( q_{1,i} = q_{i,1} = \frac{1}{n-1}, i \neq 1 \) and \( q_{i,j} = 0, i,j \neq 1 \).
  \[ \mathbb{E}(T_i) \leq \frac{1}{2\alpha-1} n \log(n). \]
  Using, direct calculation
  \[ \mathbb{E}(T_1) = \frac{1}{(2\alpha - 1)(3 - 2\alpha)} n \log(n) + O(n) \]

- **ER-graph:** \( q_{i,j} = \frac{1}{np_n} X_{i,j} X_{i,j} \) i.i.d. Bernoulli r.v. with mean \( c \frac{\log(n)}{n^2} \), \( c > \frac{2}{2\alpha-1} \), for \( h^{-1} \) the inverse of \( h(x) = x \log(x) + 1 - x \),
  \[ \mathbb{E}(T_i) \leq \frac{1}{(2\alpha - 1) h^{-1} \left( \frac{2}{c(2\alpha-1)} \right)} \log(n) + O(1) \]

- **Path:** \( \mathbb{E}(T_i) \leq \frac{16(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1) \)

- **Ring:** \( \mathbb{E}(T_i) \leq \frac{4(1-\alpha)^2}{\pi^2} n^2 \log(n) + O(1) \).
Summary

- Upper bound on the expected convergence time for a number of distributed for solving Majority consensus.

- Bounds based on the location of the spectral gap of rate matrix (generalised-cut: quick for expander graphs).

- For binary consensus, expected convergence time critically depends on the voting margin.

- Application to particular network topologies: complete graphs, stars, ER graph, paths, cycles.
At $t = 0$, each vertex of $G$ is blue independently with constant probability $\alpha \in (1/2, 1)$.

**Local Majority**

We then run $\mathcal{MP}^k$ on $G$. Choose $k$ odd ($k \geq 5$ in what follows).

- At each time $t$, each vertex $v$ polls $k$ neighbours, and assumes majority colour.
- If $v$ doesn't have $k$ neighbours, poll all, or all minus one.

What is the probability that there will be a red consensus?

How long does it take to reach consensus?
Let $V = [n]$

$\mathcal{G}_n(d)$: the set of connected simple graphs with degree sequence $d = (d_1, d_2, \ldots, d_n)$, where $d_i$ is the degree of vertex $i \in V$.

Need some restrictions on degree sequence to make it graphical, e.g., $\sum_i d_i$ is even.
Let $V_j = \{i \in V : d_i = j\}$, $= \frac{1}{n} \sum_{i=1}^n d_i$ be the average degree, $0 < \kappa \leq 1$, $0 < c < 1/8$ constants, and let $\gamma = (\sqrt{\log n})^{1/3}$. A degree sequence $d$ is nice if it satisfies

(i) Average degree $= o(\sqrt{\log n})$.

(ii) Minimum degree $\delta \geq 3$.

(iii) Let $d \geq 5$ be such that $|V_d| = \kappa n + o(n)$. We call $d$ the effective minimum degree.

(iv) Number of little vertices $\sum_{j=\delta}^{d-1} |V_j| = O(n^{1/11})$; a vertex $i$ is little if $d_i \leq d - 1$.

(v) Maximum degree $\Delta = O(n^{1/11})$.

(vi) Upper tail size $\sum_{j=\gamma}^{\Delta} n_j = O(\Delta)$. 
(iii) Let $d \geq 5$ be such that $|V_d| = \kappa n + o(n)$. We call $d$ the **effective minimum degree**.

Need not be a constant, can have $d \to \infty$ as $n \to \infty$

Not necessarily the minimum degree (though it can be)

Can have “little” vertices with smaller degree, as long as not too many of them:

(iv) Number of little vertices $\sum_{j=\delta}^{d-1} |V_j| = O(n^{\frac{1}{11}})$; a vertex $i$ is little if $d_i \leq d - 1$. 

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Examples of nice degree sequences

- Any $d$-regular graph with $d \geq 5$ and $d = o(\sqrt{\log n})$
- ‘Bi-regular’ graph where half the vertices are degree $d \geq 5$ and half of degree $\Delta = o(\sqrt{\log n})$.
- Truncated power-law
Suppose $G$ is typical with effective min degree $d$. If we run $\mathcal{MP}^k$ then

**Upper bound**

If $d/k = O(1)$ and $\alpha$ is ’not too close’ to $1/2$, then $\text{whp}$, correct consensus is reached within $(A \log_k d) \log_k \log_k n$ steps

$(A \leq 5$ and $A \to 1$ if $k \to \infty)$

**Lower bound**

Any algorithm where a vertex keeps its colour if same as all neighbours, will take at least $\log_d \log_d n$ steps to reach correct consensus, $\text{whp}$
Bias condition

“$\alpha$ is not too close to 1/2” means

\[
\left[ \left( 1 + \frac{1}{\sqrt{k}} \right)^2 \right]^{\frac{2}{k-2}} \alpha(1 - \alpha) < 1/4
\]

Since $\alpha \neq 1/2 \Rightarrow \alpha(1 - \alpha) < 1/4$, so inefficiency is in

\[
\left[ \left( 1 + \frac{1}{\sqrt{k}} \right)^2 \right]^{\frac{2}{k-2}}
\]

$k = 5$ needs $1 - \alpha < 0.143$

$k = 20$ needs $1 - \alpha < 0.350$

$k = 100$ needs $1 - \alpha < 0.437$
E. Mossel, J. Neeman, O. Tamuz (’14) Study local majority on $d$-regular $\lambda$-expanders. Show sufficient bias implies certain correct consensus.

   better bias condition but only regular graphs, no timing information, full polling only
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Y. Kanoria and A. Montanari (’10) Study local majority on $d$-regular infinite tree. Give bias conditions for convergence to majority

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J. Cruise and A. Ganesh (’10) Study $(m,d)$-generalisation of local majority on complete graphs with unit rate exponential on each vertex. Give exponential decay error probability and $O(\log n)$ timing

- stronger error probability, -only complete graph
Typical graphs: For a nice degree sequence $d$, the space $G_n(d)$ is the set of nice graphs.

We do not analyse for the whole space, only for those graphs called **typical**.

Informally, $G$ is typical if it is nice and:

- most vertices are locally tree-like
- little vertices and very high-degree vertices, should they exist, are far from each other and small cycles

Let $G'_n(d) \subset G_n(d)$ be the typical graphs, then

$$|G'_n(d)|/|G_n(d)| \to 1 \text{ as } n \to \infty$$
Let $\mathcal{T} = G[v, c \log_k \log_k n]$.
At $t + 1$, each $x \in V$ randomly picks a $x(k)$-subset of neighbours $N_x(t + 1)$

- $x \not\in \mathcal{T}$ then $x$ becomes at $t + 1$ the majority colour of the vertices in $N_x(t + 1)$.

- non-leaf $x \in \mathcal{T}$ and Par($x$) the parent of $x$ in $\mathcal{T}$. At $t + 1$, $x$ becomes the majority colour of the vertices in $N_x(t + 1)$, with the added assumption that Par($x$) was red at time $t$.
For a vertex \( v \), let \( X_v(t) \) be the indicator \( v \) is red at time \( t \) under \( \mathcal{MPP}^k \). Let \( k = 2r + 1 \).

- At time \( t = 0 \), for each level 2 (i.e., leaf) vertex \( v \),
  \[ P(X_v(0) = 0) = p_0 = 1 - \alpha \]
- At time \( t = 1 \), for each level 1 vertex \( v \)
  \[ P(X_v(1) = 0) = p_1 = P(\text{Bin}(2r, p_0) \geq r) \]
- At time \( t = 2 \), for each level 0 vertex \( v \) (i.e., the root)
  \[ P(X_v(2) = 0) = p_2 = P(\text{Bin}(2r, p_1) \geq r) \]
Modified majority protocol

If height of the tree is $H$, then given $p_t$, at $t + 1$, for $v$ at distance $H - t - 1$ from root,

$$\mathbb{P}(X_v(t + 1) = 0) = p_{t+1} = \mathbb{P}(\text{Bin}(2r, p_t) \geq r)$$

and we get a rapidly decaying sequence $p_0 > p_1 > \ldots > p_t$ with $p_0 = \alpha \gg p_t$ when $t$ large

When $t = \Omega(\log \log n)$, $p_t$ is very small and we conclude by union bound over all $n$ vertices

The root will have the correct colour.

Now we are left to deal with vertices not locally tree-like...
Theorem: Erdös-Renyi graphs

Let \( p = \frac{c \log n}{n} \) where \( c > 2 + \epsilon \) for some constant \( \epsilon > 0 \), \( k \geq 5 \) and \( \nu = \left\lfloor \frac{k-1}{2} \right\rfloor \). Run \( \mathcal{MP}^k \) on \( G \in \mathcal{G}(n, p) \).

Let \( A = \frac{1+\epsilon}{\log_k(k-1)-\log_k 2} \) where \( \epsilon > 0 \) is a small constant. Subject to condition

\[
\left[ \left( 1 + \frac{1}{\sqrt{2\nu}} \right) 2 \right]^{\frac{1}{\nu-1}} 4\alpha(1-\alpha) < 1
\]

by time \( A \log_k \log_k n \), \( \mathcal{MP}^k \) will have reached consensus on the initial majority \( \text{whp} \).
Asymptotic correct and efficient consensus using local polling. What happens for other values of $k$? [Cooper-Elsasser-Radzik’14]

Analysis for a sparse family of graphs and dense E-R graphs.

Still lot of ongoing interest...
Related work

- Jung, Kim, Vojnovic, Distributed Ranking in Networks with Limited Memory and Communication, IEEE ISIT 2012
- Mossel, Neeman and Tamuz, Majority Dynamics and Aggregation of Information in Social Networks, Autonomous Agents and Multi-Agent Systems, 2014
- Mertzios, Nikoletseas, Raptopoulos, Spirakis, Determining Majority in Networks with Local Interactions and very Small Local Memory, ICALP 2014
- Becchetti, Clementi, Natale, Pasquale, Silvestri, Trevisan, Simple dynamics for plurality consensus. SPAA 2014.