Schrödinger Bridges
classical and quantum
evolution

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• History of Schrödinger bridges
• Bridges for Markov chains
• The Hilbert metric
• Bridges for quantum (TPTP) evolutions
• Bridges for Gauss-Markov process
Schroedinger 1931/32: The time reversal of the laws of nature

Kolmogoroff: The reversibility of the statistical laws of nature

Bernstein 1932
Fortet 1940
Beurling 1960
Jamison 1974/75
Follmer 1988

connections to Nelson’s stochastic mechanics
Zambrini, Wakolbinger, Dai Pra, Pavon, Ticozzi and others
Hilbert metric:

Hilbert 1895
Birkhoff 1957
Bushell 1973

Sepulchre, Sarlette, Rouchon 2010
Reeb, Kastoryano, Wolf 2011
• Schrodinger 1931/1932: suppose a large number of Brownian particles observed at two different times to evolve between two empirical distributions. What is the most likely intermediate distribution at any point in time?
Given initial and final distribution $p_0(x)$, $p_T(x)$ and transition $p(x, y)$

Schrödinger hypothesised that

$$p_T(\cdot) \neq \int p(x, \cdot)p_0(x)dx$$

$$=: \Pi_{0T}(p_0(x))$$

$$\Pi_{0t} : q_0(x) \rightarrow q_t(x)$$

$$\frac{\partial q(t,x)}{\partial t} = \frac{1}{2} \frac{\partial^2 q(t,x)}{\partial x^2}, \quad q(0, x) = q_0(x).$$
**Schrödinger system**

- discretized time, space, N-particles
- Stirling’s approximation
- optimized, lagrange multipliers

the most likely joint density and transition probability

\begin{align*}
P^*(x_0, x_T) &= \hat{\phi}(x_0)p(x_0, x_T)\phi(x_T) \\
& \text{and} \quad p^*(x_0, x_T) = p(x_0, x_T)\frac{\phi(x_T)}{\phi(x_0)}
\end{align*}

\begin{align*}
p_0(x_0) &= \hat{\phi}(x_0) \int p(x_0, x_T)\phi(x_T)dx_T = \hat{\phi}(x_0) \underbrace{\Pi_{0,T}^{\dagger} \phi(x_T)}_{\phi_T(x_T)} \\
p_T(x_T) &= \phi(x_T) \int p(x_0, x_T)\hat{\phi}(x_0)dx_0 = \phi(x_T) \underbrace{\Pi_{0,T} \hat{\phi}(x_0)}_{\hat{\phi}_0(x_0)}
\end{align*}

and \[ p_t(x_t) = \hat{\phi}_t(x_t)\phi(x_t) \] where \[ \phi_t(x_t) := \Pi_{t,T}^{\dagger} \phi(x_T) \]
\[ \hat{\phi}_t(x_t) := \Pi_{0,t} \hat{\phi}(x_0) \]
Schrödinger system

Schrödinger: there exists a solution “except possibly for very nasty $p_0, p_T$ because the question leading to the pair of equations is so reasonable.”

Existence/uniqueness
Fortet 1940
Résolution d’un system d’équations de M. Schrödinger

Beurling 1960
An automorphism of product measures
Markov chains

\{1, \ldots, N\} states, \(x = (x_0, x_1, \ldots, x_T)\) sample path
\(\Pi_t\) stochastic matrices, \(t \in \{1, \ldots, T\}\)
\(P \in \) probability induced by \(\Pi\)’s on \(\{1, \ldots, N\}^{T+1}\)

\[P(x_0, \ldots, x_T) = P(x_0, x_T)P(x_1, \ldots, x_{T-1} \mid x_0, x_T)\]

Schrödinger question

given \(p_0, p_T\)
\(p_T \neq \Pi_T \cdots \Pi_1 p_0\)

find
\[Q(x_0, \ldots, x_T) = Q(x_0, x_T)Q(x_1, \ldots, x_{T-1} \mid x_0, x_T)\]
such that
\[\sum_{x_T} Q(x_0, x_T) = p_0(x_0)\]
\[\sum_{x_0} Q(x_0, x_T) = p_T(x_T)\]
and minimizes the relative entropy

\[
\sum_{all} Q \log \frac{Q}{P} = \sum_{x_0, x_T} Q(x_0, x_T) \log \frac{Q(x_0, x_T)}{P(x_0, x_T)} + \sum_{all} Q(\cdot \mid x_0, x_T) \log \frac{Q(\cdot \mid x_0, x_T)}{P(\cdot \mid x_0, x_T)} Q(x_0, x_T)
\]
Lagrangian

\[ L(Q) = \sum_{x_0,x_T} Q(x_0, x_T) \log \frac{Q(x_0, x_T)}{P(x_0, x_T)} \]

\[ + \sum_{x_0} \lambda(x_0) \left( \sum_{x_T} Q(x_0, x_T) - p_T(x_T) \right) \]

\[ + \sum_{x_T} \mu(x_T) \left( \sum_{x_0} Q(x_0, x_T) - p_0(x_0) \right) \]

\[ \lambda(x_0) \sim \hat{\phi}_0 \]

\[ \mu(x_T) \sim \phi_T \]

such that with \( \Pi = \Pi_T \cdots \Pi_1 \)
Schrödinger system

\[ \hat{\phi}_T = \Pi \hat{\phi}_0 \]
\[ \phi_0 = \Pi^\dagger \phi_T \]
\[ p_0 = \phi_0 \circ \hat{\phi}_0 \]
\[ p_T = \phi_T \circ \hat{\phi}_T \]

if there is a solution

\[ \Pi^* = D_{\phi_T} \Pi D_{\phi_0}^{-1} \]
\[ [Q(x_0, x_T)]_{x_0, x_T} = D_{\phi_T} \Pi D_{\hat{\phi}_0} \]
Hilbert metric

$S$ real Banach space
$K$ closed solid cone in $S$

\[ x \preceq y \iff y - x \in K, \]

\[ M(x, y) := \inf \{ \lambda \mid x \preceq \lambda y \} \]
\[ m(x, y) := \sup \{ \lambda \mid \lambda y \preceq x \}. \]

define the Hilbert metric:

\[ d_H(x, y) := \log \left( \frac{M(x, y)}{m(x, y)} \right). \]

Examples:

i) positive cone in $\mathbb{R}^n$

ii) positive definite Hermitian matrices
$d_H$-gain bound of positive maps

$\Pi$ is a positive map:

$$\Pi : K\{\{0\} \rightarrow K\{0\}.\$$

Projective diameter

$$\Delta(\Pi) := \sup\{d_H(\Pi(x), \Pi(y)) \mid x, y \in K\{0\}\}$$

Contraction ratio, or gain/$H$-norm

$$\|\Pi\|_H := \inf\{\lambda \mid d_H(\Pi(x), \Pi(y)) \leq \lambda d_H(x, y), \text{ for all } x, y \in K\{0\}\}.\$$
Birkhoff-Bushell theorem

Let $\Pi$ positive, monotone, homogeneous of degree $m$, i.e.,

$$x \preceq y \Rightarrow \Pi(x) \preceq \Pi(y),$$

and

$$\Pi(\lambda x) = \lambda^m \Pi(x),$$

then

$$\|\Pi\|_H \leq m.$$  

For the special case where $\Pi$ is also linear, the (possibly stronger) bound

$$\|\Pi\|_H = \tanh\left(\frac{1}{4}\Delta(\Pi)\right)$$

also holds.
Solution of the Schrödinger system

Lemma
Let $\Pi >_e 0$ (element-wise positive) stochastic matrix $p_0$, $p_T$ probability vectors
then $\|\Pi\|_H < 1$.

proof
i) $\Delta(\Pi) = \sup\{d_H(\Pi(x), \Pi(y)) \mid x, y \in K \setminus \{0\}\}$ remains the same if we restrict $x, y$ to be probability vectors
ii) $d_H(\Pi(x), \Pi(y)) < \infty \ \forall x, y.$
iii) the probability simplex is compact.
Solution of the Schrödinger system

Consider

\[ \hat{\varphi}_0 \xrightarrow{\Pi} \hat{\varphi}_T \]

\[ \hat{\varphi}_0(x_0) = \frac{p_0(x_0)}{\varphi_0(x_0)} \uparrow \quad \downarrow \quad \varphi_T(x_T) = \frac{p_T(x_T)}{\hat{\varphi}_T(x_T)} \]

\[ \varphi_0 \xleftarrow{\Pi^\dagger} \varphi_T \]

where

\[ D_T : \varphi \mapsto \hat{\varphi}_0(x_0) = \frac{p_0(x_0)}{\varphi_0(x_0)} \]

\[ D_T : \hat{\varphi}_T \mapsto \varphi_T(x_T) = \frac{p_T(x_N)}{\hat{\varphi}_T(x_T)} \]

are componentwise division of vectors \( \Rightarrow d_H\)-isometries!

The composition

\[ \hat{\varphi}_0 \xrightarrow{\Pi} \hat{\varphi}_T \xrightarrow{D_T} \varphi_T \xrightarrow{\Pi^\dagger} \varphi_0 \xrightarrow{D_0} (\hat{\varphi}_0)_{next} \]

is strictly contractive in the Hilbert metric.
Sinkhorn’s theorem

If $\Pi \succeq_e 0$,
then $\exists a_i, b_j$
such that $[\pi_{ij}a_i b_j]_{i,j}$ doubly stochastic.

Cf. $p_0 = 1$, $p_T = 1$
$\Pi^* = D_{\phi_T} \Pi D_{\phi_0}^{-1}$ doubly stochastic

i.e., $(\Pi^*)^\dagger 1 = 1$
but also $(\Pi^*) 1 = 1$
Quantum analogues

Density matrices: $\mathcal{D} = \{\rho \geq 0 \mid \text{trace}(\rho) = 1\}$

TPTP: $\mathcal{E}: \mathcal{D} \rightarrow \mathcal{D}: \rho \mapsto \sigma = \sum_{i=1}^{n_{\mathcal{E}}} E_i \rho E_i^\dagger$

with

$$\sum_{i=1}^{n_{\mathcal{E}}} E_i^\dagger E_i = I$$

i.e., $\mathcal{E}^\dagger(I) = I$

$\mathcal{E}$ is positivity improving: if $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) > 0$
Reference quantum evolution

TPCP maps \( \{ \mathcal{E}_t; 0 \leq t \leq T - 1 \} \)
with Kraus representation

\[
\mathcal{E}_t : \sigma_t \mapsto \sigma_{t+1} = \sum_i E_{t,i} \sigma_t E_{t,i}^\dagger, \quad t = 0, 1, \ldots, T - 1.
\]

Consider the composition

\[
\mathcal{E}_{0:T} := \mathcal{E}_{T-1} \circ \cdots \circ \mathcal{E}_0.
\]

initial and a final \( \rho_0 \) and \( \rho_T \)

Problem

Find \( \mathcal{F}_{0:T} = \mathcal{F}_{T-1} \circ \cdots \circ \mathcal{F}_0 \) such that

\[
\mathcal{F}_{0:T}(\rho_0) = \rho_T.
\]

and \( \mathcal{F} \) “close to” \( \mathcal{E} \)
“rank-1” corrections

\[ \mathcal{F}_t(\cdot) = \chi_{t+1} \left( \mathcal{E}_t(\chi_t^{-1}(\cdot)\chi_t^{-\dagger}) \right) \chi_{t+1} \]

i.e., \( \mathcal{F}_t = \Phi_{t+1} \circ \mathcal{E}_t \circ \Phi_t^{-1} \) where

\( \Phi \) are rank-1 Kraus maps, \( n_\Phi = 1 \)

Corresponds to the commutative case via: \( \chi^\dagger \chi = \phi \)
Quantum version of Sinkhorn’s thm

Suppose $\mathcal{E}_{0:T}$ is positivity improving
Then, $\exists$ observables $\phi_0$, $\phi_T$ such that,
for any factorization

$$\phi_0 = \chi_0^\dagger \chi_0, \text{ and}$$
$$\phi_T = \chi_T^\dagger \chi_T,$$

the map

$$\mathcal{F}(\cdot) := \chi_T \left( \mathcal{E}_{0:T}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^\dagger$$

is a doubly stochastic Kraus map,
in that $\mathcal{F}(I) = I$ as well as $\mathcal{F}^\dagger(I) = I$. 
Proof

\[ \hat{\phi}_0 \xrightarrow{\varepsilon_{0,T}} \hat{\phi}_T \]

\[ \hat{\phi}_0 = \phi_0^{-1} \uparrow \quad \downarrow \quad \phi_T = \hat{\phi}_T^{-1} \]

The composition map

\[ C : \left( \hat{\phi}_0 \right)_{\text{starting}} \xrightarrow{\varepsilon_{0,T}} \hat{\phi}_T \xrightarrow{\cdot} \phi_T \xrightarrow{\varepsilon_{0,T}^\dagger} \phi_0 \xrightarrow{\cdot^{-1}} \left( \hat{\phi}_0 \right)_{\text{next}} \]

is strictly contractive

the steps are identical
General case

Given $E_{0:T}^\dagger$ and $\rho_0$ and $\rho_T$
if $\exists \phi_0, \phi_T, \hat{\phi}_0, \hat{\phi}_T$ solving

\[
\begin{align*}
E_{0:T}^\dagger(\phi_T) &= \phi_0, \\
E_{0:T}(\phi_0) &= \phi_T, \\
\rho_0 &= \chi_0 \hat{\phi}_0 \chi_0^\dagger, \\
\rho_T &= \chi_T \hat{\phi}_T \chi_T^\dagger.
\end{align*}
\]

Then, for any factorization

\[
\begin{align*}
\phi_0 &= \chi_0^\dagger \chi_0, \text{ and} \\
\phi_T &= \chi_T^\dagger \chi_T,
\end{align*}
\]

the map

\[
\mathcal{F}(\cdot) := \chi_T \left( E_{0:T}(\chi_0^{-1}(\cdot)\chi_0^{-\dagger}) \right) \chi_T^\dagger
\]

is a quantum bridge for $(E_{0:T}^\dagger, \rho_0, \rho_T)$, namely $\mathcal{F}(I) = I$
and $\mathcal{F}^\dagger(\rho_0) = \rho_T$. 
Conjecture

The quantum Schrödinger system has a solution for arbitrary $\rho_0$, $\rho_T$

Snag in the proof:
$\phi \rightarrow \hat{\phi}$ and $\hat{\phi} \rightarrow \phi$ are not isometries, e.g.,

\[
D_T : \hat{\phi}_T \leftrightarrow \phi_T = \left( \rho_T^{1/2} \left( \rho_T^{-1/2} \hat{\phi}_T^{-1} \rho_T^{-1/2} \right)^{1/2} \rho_T^{1/2} \right)^2
\]

\[
\hat{D}_0 : \phi_0 \leftrightarrow \hat{\phi}_0 = (\phi_0)^{1/2} \rho(\phi_0)^{1/2}
\]

Extensive numerical evidence that the composition has a fixed point

Software for numerical experimentation

http://www.ece.umn.edu/~georgiou/papers/schrodingering.bridge/
Pinned bridge

$\mathcal{E}_{0:T}$ positivity improving and two pure states

$\rho_0 = v_0 v_0^\dagger$ and $\rho_T = v_T v_T^\dagger$

(i.e., $v_0$, $v_T$ are unit norm vectors), define

$\phi_0 := \mathcal{E}(v_T v_T^\dagger)$

$\phi_T := v_T v_T^\dagger$,

and

$\mathcal{F}^\dagger(\cdot) := \phi_T^{1/2} \mathcal{E}^\dagger(\phi_0^{-1/2}(\cdot) \phi_0^{-1/2}) \phi_T^{1/2}$

(where, clearly, $\phi_T^{1/2} = \phi_T = v_T v_T^\dagger$). Then, $\mathcal{F}^\dagger$ is TPTP and satisfies the marginal conditions

$\rho_T = \mathcal{F}^\dagger(\rho_0)$. 
Example

\[ \mathcal{E}(\cdot) = E_1(\cdot)E_1^\dagger + E_2(\cdot)E_2^\dagger + E_3(\cdot)E_3^\dagger \]

\[ E_1 = \begin{bmatrix} \sqrt{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\frac{1}{2}} \end{bmatrix}, \quad E_3 = \begin{bmatrix} 0 & \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} & 0 \end{bmatrix}. \]

\[ \rho_0 = \begin{bmatrix} 1/4 & 0 \\ 0 & 3/4 \end{bmatrix} \quad \text{and} \quad \rho_1 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix} \]

\[ \phi_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \]

\[ \phi_1 = \begin{bmatrix} 2/3 & 0 \\ 0 & 1/3 \end{bmatrix} \]

\[ \dot{\phi}_0 = \begin{bmatrix} 1/2 & 0 \\ 0 & 3/2 \end{bmatrix} \]

\[ \dot{\phi}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ F_1 = \begin{bmatrix} \sqrt{2/3} & 0 \\ 0 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{1/3} \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & \sqrt{2/3} \\ \sqrt{1/3} & 0 \end{bmatrix} \]
Recap
Hilbert metric $\Rightarrow$ constructive existence proofs for
i) classical Schrödinger systems
ii) quantum Sinkhorn version (uniform marginals)
iii) general case open

Final topic:
Schrödinger bridges for “degenerate” classical linear stochastic systems
≡ a new type of optimal control problem
Optimal steering of state-densities

\[
\min \text{ relative entropy} \leftrightarrow \text{ minimum energy stochastic control}
\]

\[
dx = bdt + dw \text{ diffusion}
\]

\[
dx = (b + u)dt + dw \text{ controlled diffusion}
\]

\[
\min \{ E\{\|u\|^2 \} \mid p_0, p_T \} \sim \text{ relative entropy from prior (dai Pra)}
\]

our interest:

inertial particles, cooling of oscillators

\[
dx = vdt
\]

\[
dv = (b + u)dt + dw \text{ controlled degenerate diffusion}
\]
Optimal steering of state-densities

\[ dx(t) = A(t)x(t)dt + B(t)u(t)dt + B(t)dw(t) \]

Given initial and terminal (target) Gaussian densities with covariances \( \Sigma_0, \Sigma_T \).

Find \( u(t) \) with \( t \in [0, T] \) that steers the system from the initial to the target state density and minimizes

\[ E\{ \int_0^T u(t)'u(t)dt \} \]
Optimal steering of state-densities

Theorem (Gauss-Markov Schrödinger bridge): There exists a unique solution to the following (analogue of the Schrödinger system)

\[ Q(T), \, P(0) \text{ values for matrices satisfying} \]

\[ \Sigma_0^{-1} = Q(0)^{-1} + P(0)^{-1} \]
\[ \Sigma_T^{-1} = Q(T)^{-1} + P(T)^{-1} \]

and \( Q(0), \, P(T) \) obtained via

\[ \dot{Q}(t) = A(t)Q(t) + Q(t)A(t)' + B(t)B(t)' \]
\[ \dot{P}(t) = A(t)P(t) + P(t)A(t)' - B(t)B(t)' \]

with \( Q(t) \) invertible over \([0, T]\).
The optimal control is $u(t) = -B(t)'Q(t)^{-1}x(t)$
The controlled degenerate diffusion is the closest to the uncontrolled diffusion in the relative entropy sense.

\[
Q(0) = N(T, 0)^{1/2} S_0^{1/2} \left( S_0 + \frac{1}{2} I - \left( S_0^{1/2} S_T S_0^{1/2} + \frac{1}{4} I \right)^{1/2} \right)^{-1} S_0^{1/2} N(T, 0)^{1/2}
\]

$N(T, 0)$ is the controllability Grammian.
Gauss Markov model for inertial particles

\[ dx(t) = v(t) \, dt \]
\[ dv(t) = u(t) \, dt + dw(t) \]
Gauss Markov model for inertial particles
Gauss Markov model for Nyquist-Johnson noise driven oscillator

\[
Ldi_L(t) = v_C(t)dt \\
RCdv_C(t) = -v_C(t)dt - Ri_L(t)dt + u(t)dt + dw(t)
\]
Gauss Markov model for inertial particles: state-cost \sim particles with losses

\[ dX(t) = f(X(t), t)dt + \sigma(X(t), t)dw(t) \]

\[
\inf_{(\tilde{\rho}, \tilde{u})} \int_{\mathbb{R}^N} \int_0^T \left[ \frac{1}{2} \|u\|^2 + V(x, t) \right] \tilde{\rho}(x, t) dt dx,
\]

\[
\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot ((f + \sigma u) \tilde{\rho}) = \frac{1}{2} \sum_{i,j=1}^N \frac{\partial^2 (a_{ij} \tilde{\rho})}{\partial x_i \partial x_j},
\]

\[ a_{ij}(x, t) = \sum_k \sigma_{ik}(x, t) \sigma_{kj}(x, t) \]

\[
\tilde{\rho}(0, x) = \rho_0(x), \quad \tilde{\rho}(T, y) = \rho_T(y). \]
Schrödinger system

\[
\frac{\partial \varphi}{\partial t} + f(x, t) \cdot \nabla \varphi + \frac{1}{2} \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 \varphi}{\partial x_i \partial x_j} = V \varphi,
\]

\[
\frac{\partial \hat{\varphi}}{\partial t} + \nabla \cdot (f(x, t) \hat{\varphi}) - \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 (a_{ij} \hat{\varphi})}{\partial x_i \partial x_j} = -V \hat{\varphi},
\]

\[
\varphi(x, 0)\hat{\varphi}(x, 0) = \tilde{\rho}_0(x), \quad \varphi(x, T)\hat{\varphi}(x, T) = \tilde{\rho}_T(x)
\]

\[
u^*(x, t) = \sigma' \nabla \log \varphi(x, t),
\]

\[
\frac{\partial \tilde{\rho}}{\partial t} + \nabla \cdot ((f + a \nabla \log \varphi) \tilde{\rho}) = \frac{1}{2} \sum_{i,j=1}^{N} \frac{\partial^2 (a_{ij} \tilde{\rho})}{\partial x_i \partial x_j},
\]
Controllability of Fokker-Planck - Linear-Gaussian

\[ dx(t) = Ax(t)dt + Bu(t)dt + B_1dw(t) \]
with \( x(0) = x_0 \) a.s.

Thm: (A,B) controllable is sufficient to steer the system from any initial Gaussian distribution to a final one at \( t=T \).

Thm: A Gaussian state-pdf can be “sustained” with constant state-feedback iff the state covariance satisfies

\[ (A - BK)\Sigma + \Sigma(A' - K'B') + B_1B'_1 = 0. \]

equivalently,

\[ \text{rank} \begin{bmatrix} A\Sigma + \Sigma A' + B_1B'_1 & B \\ B & 0 \end{bmatrix} = \text{rank} \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \]
Compare with conditions for:

i) steering the system to a given state - controllability
ii) steering within the positive cone?
iii) maintaining the state at a given value

\[ \dot{x}(t) = Ax(t) + Bu(t) \]

\[ 0 = (A - BK)\xi + Bu \]
Schrödinger system

\[ \dot{\Pi} = -A'\Pi - \Pi A + \Pi B B' \Pi \]
\[ \dot{H} = -A'H - HA - HBB'H + (\Pi + H) (BB' - B_1B_1') (\Pi + H) \]
\[ \Sigma_0^{-1} = \Pi(0) + H(0) \]
\[ \Sigma_T^{-1} = \Pi(T) + H(T). \]
Fast “cooling” + stationary control

& for dw anywhere
Open problem

Density matrices: e.g. $D = \{ \rho \geq 0 \mid \text{symmetric } \rho \in \mathbb{R}^{n \times n} \text{ with trace}(\rho) = 1 \}$

$E_i$ with $i = 1, \ldots, n_E$ and $\sum_{i=1}^{n_E} E_i^\dagger E_i = I$
(typically $n_E \sim n^2$

for “positivity-improving”: $\rho \geq 0 \Rightarrow \mathcal{E}(\rho) > 0$)

TPTP: $\mathcal{E} : \mathfrak{D} \rightarrow \mathfrak{D} : \rho \longrightarrow \sigma = \sum_{i=1}^{n_E} E_i \rho E_i^\dagger$

Data: $\rho_0, \rho_T, \mathcal{E}$.

Problem: Prove that the iteration:

$\mathcal{E} : \hat{\phi}_0 \mapsto \hat{\phi}_T = \mathcal{E}(\hat{\phi}_0)$

$D_T : \hat{\phi}_T \mapsto \phi_T = \left( \rho_T^{1/2} \left( \rho_T^{-1/2} \hat{\phi}_T^{-1/2} \rho_T^{-1/2} \right)^{1/2} \rho_T^{1/2} \right)^2$

$\mathcal{E}^\dagger : \phi_T \mapsto \hat{\phi}_0 = \mathcal{E}^\dagger(\phi_T)$

$\hat{D}_0 : \phi_0 \mapsto \hat{\phi}_0 = (\phi_0)^{1/2} \rho(\phi_0)^{1/2}$

has an attractive fixed point.

Software for numerical experimentation

http://www.ece.umn.edu/~georgiou/papers/schrodinger_bridge/
Thank you for your attention
http://arxiv.org/abs/1405.6650
Positive contraction mappings for classical and quantum Schrödinger systems

http://arxiv.org/abs/1407.3421
Stochastic bridges of linear systems

Optimal steering of inertial particles diffusing anisotropically with losses

arxiv.org/abs/1408.2222
Optimal steering of a linear stochastic system to a final probability distribution

arxiv.org/abs/1410.3447
Optimal steering of a linear stochastic system to a final probability distribution, Part II