Synchronization of oscillators: Feasability and Non-local analysis

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1 System Definition

We consider a network of n oscillators each characterized by a phase angle $\theta_i \in \mathbb{S}^1$ (where $\mathbb{S}$ denotes the unit circle) and a distinct natural frequency $\omega_i \in \mathbb{R}$. The oscillator phases evolve according to

$$\frac{d}{dt}\theta_i(t) = \omega_i + \sum\nolimits_{j=1}^{n} a_{ij}\sin\left(\theta_j(t) - \theta_i(t)\right), \tag{1}$$

where $a_{ij} = a_{ji} \geq 0$ are symmetric and non-negative coupling weights. The oscillator interactions are represented by an undirected, connected, and weighted graph $G(\{1,\ldots,n\}, E, A)$ induced by the matrix with elements a_{ij}. A particular instance of model (1) is the well known Kuramoto model [9] in which the graph is complete and the coupling is uniform: $a_{ij} = K/n$ for all i,j and some $K > 0$. We emphasize that we are interested in heterogeneous natural frequencies since otherwise (1) reduces to a mere gradient flow.

A long-term objective in the study of systems of the form (1) is to understand the conditions under which frequency synchronization takes place, that is, under which all frequencies $\dot\theta_i$ eventually converge to a same value. A large number of particular results are available on this topic [1, 2, 6, 11], but the general problem is considered to be very challenging, and these systems are known to sometimes exhibit surprisingly complex behaviors. We propose here two open problems whose solution could represent major breakthroughs in the domain as well as in applications. We first need to introduce some additional notions.

Notations, Conventions, and Preliminaries

An angle angle is a point on the unit circle $\mathbb{S}^1$. An arc is a connected subset of $\mathbb{S}^1$. For two angles $\theta_1, \theta_2 \in \mathbb{S}^1$, we denote by $|\theta_2 - \theta_1|$ the geodesic distance between these two angles, that is, the minimum of the clockwise and counterclockwise arc length connecting θ_1 to θ_2. The *angular difference* $\theta_2 - \theta_1$ is the number in $]-\pi, \pi]$ whose magnitude is equal to $|\theta_2 - \theta_1|$, with positive sign if and only if the counter-clockwise path length from θ_1 to θ_2 is smaller than the clockwise path length.

A (frequency-)*synchronized* trajectory of (1) is one for which $\dot\theta_i(t) = \omega_{\text{sync}}$ for every i and some $\omega_{\text{sync}} \in \mathbb{R}$. Due to the symmetry of the coupling terms $a_{ij}\sin(\theta_i - \theta_j)$, one can easily verify that $\sum_{i=1}^{n} \dot\theta_i(t) = \sum_{i=1}^{n} \omega_i$ for all t. Therefore, $\omega_{\text{sync}} = \frac{1}{n}\sum_{i=1}^{n}\omega_i$ holds for every synchronized trajectory. Since adding a constant value to all ω_i does not affect the synchronization properties of the system, we assume without loss of generality that $\frac{1}{n}\sum_{i=1}^{n}\omega_i = 0$ and thus that $\omega_{\text{sync}} = 0$. In particular, synchronized trajectories are then constant and their analysis reduces to the study of equilibria of (1).

Observe that system (1) is also invariant under a rigid rotation of all angles, in the sense that if σ and θ are two trajectories for which the initial conditions are related by rotation, $\sigma_i(0) = \text{rot}_\delta(\theta_i(0))$ for every i, and some angle $\delta \in \mathbb{S}^1$, then $\sigma_i(t) = \text{rot}_\delta(\theta_i(t))$ holds for all i, t. This equality up to a constant phase difference defines an equivalence relation on the trajectories.

By an abuse of language, we then say that two trajectories σ, θ approach each other if $\lim_{t\to\infty} \theta_i(t) - \sigma_i(t) = \delta$ for some constant phase difference δ. We then say that a trajectory θ is *locally attractive* if there exists an open set $\Theta \in \mathbb{T}^n$ (where $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$) containing $\theta(0)$ such that every trajectory σ for which $\sigma(0) \in \Theta$ approaches θ. This set Θ is called the attraction basin. Similarly, the trajectory is (almost) globally attractive if trajectories starting at (almost) every initial condition approach θ.

2 Question 1: Existence & Local Stability of Synchronization

The following question is posed in every survey article on synchronization of coupled oscillators [1,2,6,11]:

Question 1. *Given an undirected, connected, and weighted graph $G(\{1,\ldots,n\}, E, A)$ and natural frequencies $\omega \in \mathbb{R}^n$ satisfying $\sum_{i=1}^n \omega_i = 0$, find a closed form criterion to decide if (a) there exists a synchronized trajectory, (b) there exists a locally attractive synchronized trajectory.*

Recall that a trajectory is synchronous if and only if it satisfies $\dot{\theta}_i = \omega_{\text{sync}} = 0$ for every i, and thus

$$\omega_i = \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j), \qquad \forall i \in \{1, \ldots, n\}. \tag{2}$$

Question 1(a) is equivalent to the existence of a solution $\theta^* \in \mathbb{T}^n$ to the set of algebraic equations (2). It has proved surprisingly difficult to analyze. Question 1(b) is usually addressed for equilibria satisfying $|\theta_i^* - \theta_j^*| < \pi/2$ for all $\{i,j\} \in E$. In this case, the negative Jacobian of (1) is a Laplacian with nonnegative weights $a_{ij} \cos(\theta_i^* - \theta_j^*)$. The local stability question is far from trivial for other classes of equilibria.

Question 1 is not easy to analyze, but several necessary or sufficient conditions have been provided. For example, $\sum_{j=1}^n a_{ij} \geq \omega_i$ is obviously necessary for the equations (2) to admit a solution. Many of the necessary conditions available in the literature are derived from variations of this inequality [6]. Stronger results are available in particular cases, e.g., for the densest (complete) and sparsest (acyclic) graphs.

For the Kuramoto model $a_{ij} = K/n$ on a complete graph and for natural frequencies in a range $[\omega_{\min}, \omega_{\max}]$, the simple necessary condition can be reformulated to $K > (\omega_{\max} - \omega_{\min})n/(2(n-1))$ [6]. A sufficient and tight condition is $K > \omega_{\max} - \omega_{\min}$ [5], which is a factor $n/(2(n-1))$ from the necessary condition. An exact condition can be obtained as solution to two implicit nonlinear equations [12].

The second case concerns acyclic networks: Let L be the Laplacian matrix of the graph defined by $L_{ij} = -a_{ij}$ if $i \neq j$ and $L_{ii} = \sum_{j=1}^n a_{ij}$. When G is acyclic, there exists a locally stable synchronous trajectory if and only if $\max_{(i,j)\in E} |[L^\dagger \omega]_i - [L^\dagger \omega]_i| < 1$, where $L^\dagger$ is the Moore-Penrose pseudo-inverse of L [7]. This condition appears to be also sufficient for various other topologies. Extensive Monte Carlo simulations on randomized network topologies, weights, and natural frequencies have shown that $\max_{(i,j)\in E} |[L^\dagger \omega]_i - [L^\dagger \omega]_j| \geq 1$ is almost always a sufficient and nearly tight condition. However, rare counterexamples have been observed, and the condition is in general neither necessary nor sufficient.

3 Question 2: Non-Local Stability Analysis

Question 2. *Prove, for some nontrivial class of interactions weights a_{ij} and natural frequencies ω_i*

- *(strong version) the existence of an almost globally attractive frequency-synchronized trajectory.*

- *(weak version) the existence of a semiglobally attractive frequency-synchronized trajectory whose basin of attraction Θ is spread on a significant part of the state space $\mathbb{S}^n$, that is, a basin of attraction **not** included in any of the two following sets:*

  $\mathcal{B} := \{\theta \in \mathbb{S}^n : (i,j) \in E \Rightarrow |\theta_i - \theta_j| < \pi/2\}$
  $\mathcal{C} := \{\theta \in \mathbb{S}^n : \exists \delta : \theta_i \in [\delta, \pi + \delta[, \forall i\}$

Almost all non-local results on synchronization available in the literature involve basins of attraction that are subsets of the sets $\mathcal{B}$ or $\mathcal{C}$, see [6] for a review. For homogeneous natural frequencies, there are certain notable exceptions, such as the almost global convergence results for the Kuramoto model [10].

In the set $\mathcal{B}$ or $\mathcal{C}$ the problem becomes more tractable by "cutting the circle manifold into an interval". For example, when $\theta \in \mathcal{B}$ the system (1) can be seen as a linear consensus (with time-varying weights) on the frequencies. Indeed, observe that the frequencies $x_i(t) = \dot{\theta}_i(t)$ satisfy

$$\frac{d}{dt}x_i(t) = \sum_{j=1}^{n} \tilde{a}_{ij}(t)(x_j(t) - x_i(t)), \tag{3}$$

with $\tilde{a}_{ij}(t) = a_{ij}\cos(\theta_i(t) - \theta_j(t))$. Therefore, as long as $\theta(t) \in \mathcal{B}$, the weights $\tilde{a}_{ij}(t)$ are non-negative, and system (3) is a contracting consensus system so that all $x_i(t) = \dot{\theta}_i(t)$ synchronize. Thus, local stability of a synchronized trajectory can be assured if the set $\mathcal{B}$ can be shown to be forward invariant for system (1).

Similarly, observe that as long as $\theta \in \mathcal{C}$, the angles θ_i can be mapped onto real numbers $\tilde{\theta}_i$ evolving as

$$\frac{d}{dt}\tilde{\theta}_i(t) = \omega_i + \sum_{j=1}^{n} \tilde{b}_{ij}(t)\left(\tilde{\theta}_j(t) - \tilde{\theta}_i(t)\right), \tag{4}$$

with $\tilde{b}_{ij}(t) = a_{ij}\frac{\sin(\tilde{\theta}_j - \tilde{\theta}_i)}{\tilde{\theta}_j - \tilde{\theta}_i} \geq 0$ being strictly positive if and only if a_{ij} is positive. One approach to establish local stability is again to find conditions under which the system (4) is well defined on the reals and converges. The requirement that system (4) is well defined on the reals is guaranteed if $\tilde{\theta}(t) \in \mathcal{C}$.

In the above two cases, it is thus possible to perform an analysis similar to one performed on a system defined on reals. Therefore, we think that finding attraction basins not included in $\mathcal{B}$ or $\mathcal{C}$ represents an important challenge, because it requires departing from systems defined on reals, and taking into account the complexity induced by the fact that the phases live on the circle manifold $\mathbb{S}$. It would thus represent an important breakthrough in the analysis of oscillator networks.

To conclude this section, we mention two more specific conjectures that would answer Question 2. Existence and local stability is well understood for complete and acyclic graphs, see the conditions $K > \omega_{\max} - \omega_{\min}$ and in Section 2. We conjecture almost global stability in both cases.

Conjecture 1 (Kuramoto model). *Consider the Kuramoto model with $a_{ij} = K/n$ for all i, j for some K. If there exists a locally asymptotically stable synchronized trajectory, then it is almost globally attractive.*

Conjecture 2 (Acyclic topology). *Consider the coupled oscillator model on an acyclic graph G. If there exists a locally asymptotically stable synchronized trajectory, then it is almost globally attractive.*

4 Relevance to Engineering Applications

The above questions are particularly relevant in the context of power network applications. The steady-state equations (2) of the coupled oscillator model (1) are the active power flow equations in a transmission network. A dynamic model for a multi-machine power system is given by structure-preserving model [3]

$$
\begin{aligned}
M_i\ddot{\theta}_i(t) + D_i\theta_i(t) &= P_i + \sum_{j=1}^{n} b_{ij}\sin(\theta_j(t) - \theta_i(t)), \quad i \in \{\text{generators}\} \\
D_i\theta_i(t) &= P_i + \sum_{j=1}^{n} b_{ij}\sin(\theta_j(t) - \theta_i(t)), \quad i \in \{\text{loads}\}
\end{aligned}
\tag{5}
$$

where the state variables θ_i are the generator rotor angles (at generator buses) and the voltage phase angles (at load buses), respectively. The inertia and dissipation constant for a generator bus are M_i and $D_i > 0$, for a load bus $D_i > 0$ is load-frequency coefficient, $P_i \in \mathbb{R}$ are the active power injections (positive for generators and negative for loads), and b_{ij} are the network susceptances.

Question 1(a) is directly concerned with feasibility of the power flow equations which is of tremendous importance for all of power system studies, e.g., feasibility of power flow optimization problems. Question 1(b) concerns the non-singularity and stability of the load flow Jacobian which is of importance in distance to failure-metrics and numerical algorithms [8]. Question 2 is known as transient stability problem in power systems, and it is generally considered to be the "holy grail" of power system stability studies [4]. We remark that any local stability result found for a synchronized trajectory first-order coupled oscillator system (1) immediately implies existence and local stability of the same trajectory for the mixed first-/second-order system (5), though the basins of attraction may be very different in the two cases, see [5].

Thus, the considered questions are of tremendous importance and have immediate applications in power network problems. When the coupling terms $a_{ij} \sin(\theta_i - \theta_j)$ in the coupled oscillator model (1) are slightly modified to include asymmetric interactions and higher order harmonics, then other immediate applications of coupled oscillators are in biology, neuroscience, and robotic coordination [1, 2, 6, 11].

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